Thermal AdS_3, BTZ and competing winding modes condensation

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ABSTRACT: We study the thermal physics of AdS_3 and the BTZ black hole when embedded in String theory. The exact calculation of the Hagedorn temperature in $TAdS_3$ is reinterpreted as the appearance of a winding tachyon both in AdS_3 and BTZ. We construct a dual framework for analyzing the phases of the system. In this dual framework, tachyon condensation and geometric capping appear on the same footing, bridging the usual gap of connecting tachyon condensation to modifications of geometry. This allows us to construct in a natural way a candidate for the unstable phase, analogous to a small black hole in higher dimensions. Additional peculiar effects associated with the Hagedorn temperature and the Hawking-Page transition, some to do with the asymptotic structure of AdS_3 and some with strong curvature effects, are analyzed and explained.

Contents

1. Introduction and summary of results

The thermodynamical interpretation of classical gravity in asymptotically AdS_d spaces [1 – 4] predicts a maximal temperature for a gas of thermally excited strings (the Hawking-Page temperature [5]). At temperatures higher than the HP temperature the gas of strings collapses into a black hole via a first order phase transition. If we keep the gas of strings in an overheated metastable phase (preventing the HP phase transition) while raising the temperature, we will eventually reach the Hagedorn temperature where the gas of strings must collapse into a black hole because the barrier disappears and a tachyonic mode appears [6 – 9]. This tachyon is similar to the Atick-Witten tachyon [10] in flat space, which is associated with the Hagedorn temperature for that configuration.

In this paper we explore both the Atick-Witten and Hagedorn phase transitions in asymptotically AdS_3 spaces using the exact, in α', worldsheet description (for the most

part). For both thermal AdS_3 and Euclidean BTZ the boundary of spacetime looks like a radial direction$\times S_t^1 \times S_\theta^1$ where t is the Euclidean time direction and θ an angular variable. In $TAdS_3$ the radial direction and S_θ^1 combine to form a 2-dimensional disk and in the Euclidean BTZ the radial direction pairs up with S_t^1. Above the Hagedorn temperature a winding mode tachyon appears around the t direction. It is generally believed, and has been argued in various ways (for example [6–9, 12–14]) that such a tachyon causes the t circle to pinch, changing the topology of the background to that of BTZ, to which the system then relaxes.

This is a compelling scenario, but following this topology change in details is rather complicated, as one needs to follow the flow of the worldsheet through this topology change in which the tachyon mixes with, or induces, metric deformations. What is sometimes done for this case is either to argue the effects of the tachyon based on general worldsheet RG properties, or to describe the two topologies using a Ricci flow, which either ends or starts from a singularity - when the topology change occurs - and to glue the two flows at the singularity in a somewhat ad-hoc (although correct) manner [15].

There are many works on closed string tachyon condensation. It is common to distinguish localized closed string tachyons and bulk closed string tachyons. The former have a relatively mild effect as they deform the geometry in the vicinity of the region in which they are localized, usually capping the geometry and making a small part of space disappear. This process may be followed by emission of some perturbative massless and massive bulk modes. For references on the subject of closed string tachyon condensation see [9, 11, 16, 17] and references therein.[1] Bulk tachyons, on the other hand, are much harder to understand, and their condensation is expected to reduce the number of space time dimensions [23], as a consequence of the Zamolodchikov c-theorem [22]. The tachyons that are relevant for phase transitions in AdS spaces are either localized tachyons or de-localized for AdS_3 with NS-NS fields (which nevertheless have effects similar to localized tachyons).

In this paper we take some steps towards improving the understanding of the flow around the topology change point. The immediate context which we will discuss is the Hawking-Page and Hagedorn phase transitions in AdS_3, where we can rely on the notion of a dual CFT, but we expect that a similar set of tools will be useful to discuss topology change in more general cases. The main idea is to convert the background geometric data into a tachyon condensation problem, like in the FZZ duality [24]. The problem then reduces to a simpler problem of comparing the strength of the "geometric" tachyon wall with the strength of the new Atick-Witten-like tachyonic wall. This can be made very precise in the case of $TAdS_3/BTZ$ which is what we will do next.

Strings propagating in AdS_3 are described by the SL$(2,\mathbb{R})$ WZW model. To describe the thermal gas of strings we use the Euclidean version of the CFT (the H_3^+ model) and compactify the Euclidean time coordinate. By a sequence of T-dualities we bring this space

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[1]For a review of tachyon condensation in a cosmological context see [18] and [19]. Phase transitions to bubbles of nothing in AdS_5 which are driven by winding tachyons were discussed in [20, 21].

to the form of a $cigar \times S^1$ which is simply the coset

$$\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{U}(1)} \times \mathrm{U}(1)$$

(with some important gluing conditions). By applying an FZZ duality the theory is mapped into a $sine - Liouville \times S^1$, which is a $linear\ dilaton \times S^1 \times S^1$ with an interaction term coupling the Liouville mode (ϕ) with a winding mode around the circle related to a spatial angle x which is roughly the angular direction of AdS_3,

$$e^{b\phi} \cos R_x(x_L - x_R).$$

On the other hand, the Atick-Witten thermal tachyon is a winding mode around another circle φ, which is roughly the Euclidean time direction t, and it is of the form

$$e^{b'\phi} \cos R_\varphi(\varphi_L - \varphi_R)$$

This achieves the goal of putting both metric and tachyon deformations on the same footing in a manifest way, which simplifies considerably the analysis of the topology change point. For example, the phase transition between the thermal gas and black hole is a competition (by RG flow) between two sine-Liouville interaction terms, both including winding modes but on different circles. More generally, this method provides us with a "unified" description of the various backgrounds in which phases and phase transitions of different kinds are treated similarly. These include the Hawking Page phase transition, the Hagedorn phase transition and the fixed points that exist between them.

At the end of the day, this dynamics is controlled by computing dimensions of some specific operators in sine-Liouville theory (which we review in section 4.1). In the case that these operators are irrelevant, they drive a geometric capping (and a new CFT) by turning them on with a large coefficient which corresponds to a first order phase transition in spacetime. The other case, when they are marginal or relevant, is the case beyond the Hagedorn temperature, where their condensation is exactly the Atick-Witten tachyon condensation.

Another result of our investigations is the surprising understanding that thermal AdS_3 and the BTZ black hole at the same temperature have a pathological canonical thermodynamic descriptions if embedded in String theory (unlike the case in pure gravity). However, they are expected to have well defined micro-canonical description.

The paper is organized as follows: In section 2 we briefly review known results concerning String theory on thermal AdS_3. section 3 is devoted to the calculation of the Hagedorn temperature for AdS_3, clarification of the phase diagram and discussion of some special phenomena which take place at strong curvature. section 4 and section 5 contain the main results. In section 4 we explain the mapping of the thermal theory into a $cigar \times S^1$. Then, applying FZZ duality, we map the theory into $sine - Liouville \times S^1$ where it is manifest that String theory treats equally geometric and tachyonic capping. section 5 contains a discussion of the properties of the unstable fixed point (CFT) which separates the black hole and thermal AdS phases, as well as a discussion of the possible flows in the system.

We conclude in section 6 with a discussion of open questions and propose directions for future research.

We would like to thank Shiraz Minwalla and Vadim Shpitalnik for collaboration at early stages of this work. After this work was completed, we received a draft of [25] which overlaps with some of the results in section 3, and [26] which discusses the Lorentizan BTZ.

2. Review

2.1 Solutions of Euclidean gravity

The Euclidean AdS_3 manifold is the 3 dimensional hyperbolic space H_3^+ (which is the analytic continuation of $\mathrm{SL}(2,\mathbb{R})$, and can be represented as $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$), i.e the space of Hermitian unimodular matrices

$$X = \begin{pmatrix} X_{-1} + X_1 & X_2 + iX_3 \\ X_2 - iX_3 & X_{-1} - X_1 \end{pmatrix} \qquad X_i \in \mathbb{R} \qquad \det X = 1. \qquad (2.1)$$

The manifold has an $\mathrm{SL}(2,\mathbb{C})$ isometry group acting on it as $X \to AXA^+$ where $A \in \mathrm{SL}(2,\mathbb{C})$. Throughout the paper we use global coordinates for AdS_3

$$X = e^{\frac{i}{2}(-it+\theta)\sigma^2} e^{\rho\sigma^3} e^{\frac{i}{2}(-it-\theta)\sigma^2}, \qquad (2.2)$$

where the σ^i are Pauli matrices. The line element is given by the square of the Maurer-Cartan form

$$ds^2 = \frac{k}{2} \operatorname{Tr}\left(X^{-1}dX\right)^2 = k\left(\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\theta^2\right), \qquad (2.3)$$

where k is the AdS_3 radius squared.[2] In the Euclidean theory, one can make the coordinates t and θ periodic,

$$\theta \cong \theta + 2\pi \qquad t + i\theta \cong t + i\theta - 2\pi i\tau \quad ; \qquad \tau \equiv i\frac{\beta}{2\pi}\left(1 + i\mu\right), \qquad (2.4)$$

such that at the conformal boundary $\rho \to \infty$, the geometry is a $\mathbb{T}^2$ with modular parameter τ. We use the name thermal AdS_3 (or $TAdS_3$) for such a geometry with finite τ. The parameters β and μ in (2.4) are correspondingly the inverse temperature and chemical potential of the canonical thermodynamic ensemble.

The H_3^+ WZW CFT describing strings propagating in $TAdS_3$ includes a constant dilaton (with arbitrary value) and an imaginary B-field,

$$H_{(3)} = dB_{(2)} = -2ik \sinh(2\rho) d\rho \wedge dt \wedge d\theta. \qquad (2.5)$$

The B-field is imaginary due to the analytic continuation from the Lorentzian geometry where the B-field is real. The Euclidean worldsheet action, on the other hand, is real due to the additional factor of i from the analytic rotation of the worldsheet time.

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[2]throughout the paper we use units in which $\alpha' = 1$.

In gravity, the canonical ensemble contains all Euclidean gravity solutions which have the same boundary geometry (including H-field) up to diffeomorphisms. To find the classical GR solutions contributing to the ensemble it is convenient to start from the Lorentzian AdS_3 (the $SL(2,\mathbb{R})$ group manifold) and construct new gravity solutions by orbifolding with elements of the isometry group. The interesting orbifolds are classified by conjugacy classes of $SL(2,\mathbb{R})$, of which there are three types[3] which generate three distinct types of orbifolds. The $\mathbb{Z}$ orbifolds generated by elements in the hyperbolic class are the BTZ black holes geometries. These are the only orbifolds which have a smooth Euclidean continuation with finite parameter τ. The various orbifolds (by elements of $SL(2,\mathbb{R})$) form a complete classification of solutions to 2+1 dim GR with constant negative curvature [27, 28].

The Euclidean BTZ ($EBTZ$) solutions (which are $\mathbb{Z}$ orbifolds of H_3^+) are commonly expressed in Schwarzschild coordinates [27, 28] with periodicities (2.4),

$$ds^2 = N^2 dt^2 + N^{-2} dr^2 + r^2 \left(d\theta + N_\theta dt\right)^2$$
$$N^2 = \frac{(r^2 - r_+^2)(r^2 + r_-^2)}{k^2\, r^2} \qquad N_\theta = \frac{r_+ r_-}{r^2}$$
$$r_+ = -k\, \mathrm{Im}\, \frac{1}{\tau} \qquad\qquad r_- = k\, \mathrm{Re}\, \frac{1}{\tau}.$$

The modular parameter τ is related to the Lorentizan BTZ black hole mass and angular momentum via,

$$M = \frac{r_+^2 + r_-^2}{k^2} = \left|\frac{1}{\tau}\right|^2 \qquad\qquad J = \frac{2r_+ r_-}{k} = -\frac{k}{2} \mathrm{Im}\left(\frac{1}{\tau^2} - \frac{1}{\bar{\tau}^2}\right). \qquad (2.6)$$

Maldacena and Strominger [29] argued that, starting from a $TAdS_3$ with modular parameter τ, there exists an $SL(2,\mathbb{Z})$ family of solutions[4] which at infinity have the modular parameter τ. The construction is as follows - start with the $\mathbb{H}_3^+$ manifold (Euclidean AdS_3) orbifolded by a $SL(2,\mathbb{C})$ elements of the isometry group, generating a $TAdS_3$ solution with modular parameters τ. The $SL(2,\mathbb{C})$ element generating this orbifold is

$$H = \begin{pmatrix} e^{i\pi\tau} & 0 \\ 0 & e^{-i\pi\tau} \end{pmatrix}.$$

Consider another element of $SL(2,\mathbb{C})$ generating $TAdS_3$ with conformal parameter τ' such that there is an $SL(2,\mathbb{Z})$ transformation connecting these two modular parameters,

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad ; \qquad\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \qquad (2.7)$$

Then, there exists a coordinate transformation which acts on all coordinates, but in particular it performs an $SL(2,\mathbb{Z})$ transformation near the boundary, changing the modular

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[3]The number of conjugacy classes is infinite and labeled by the trace of the $SL(2,\mathbb{R})$ matrix. It is convenient to divide them to hyperbolic, parabolic, and elliptic $Tr(M) >=< 2$ respectively.

[4]Which can perhaps also be constructed by using the technique described in this paper of applying different sine-Liouville caps on different cycles of the $\mathbb{T}^2$.

parameter from τ' to τ. Thus, for a given modular parameter τ which defines the ensemble, there is a family of Euclidean solutions corresponding to elements of $\mathrm{SL}(2,\mathbb{Z})$. These solutions are part of the same thermal ensemble. It is known that $TAdS_3$ with parameter τ is diffeomorphic to the $EBTZ$ black hole with periodicity $\tau' = -\frac{1}{\tau}$. This correspondence is a special case of the construction outlined above. The nice feature about the latter transformation is that it has a well defined continuation to Lorentzian signature. The Euclidean action of the instantons was calculated in [29],

$$S_{\text{instanton}} = \frac{i\pi k}{2}\left(\tau' - \bar{\tau}'\right) = -\frac{2\pi^2 k\beta}{c^2\beta^2 + (c\beta\mu - 2\pi d)^2}.\qquad(2.8)$$

Note that the action depends only on two of the three independent paraments of $\mathrm{SL}(2,\mathbb{Z})$, indeed the transformation $\tau' = \tau + b$ does not change the geometry or the entropy.

At each value of τ there is a dominant phase given by the lowest (negative) action instanton. At low temperature, $\beta \to \infty$, the dominant saddle point is the thermal AdS_3 solution (i.e a gas of cold particles). At very high temperature, $\beta \to 0$ the dominant saddle point has $\tau' = -\frac{1}{\tau}$, which is diffeomorphic to the $EBTZ$ black hole with this (high) temperature $1/\beta$ and chemical potential μ. In the generic case there are various phases that can dominate the path integral at intermediate value of the temperature [30, 31].

The simplest situation occurs for vanishing chemical potential $\mu = 0$, which is what we assume from now on in this paper. The simplifications is that there are only two dominating phases, which are the thermal AdS_3 at low temperature and the $EBTZ$ black hole at high-temperature. The phase transition occurs at the Hawking-Page temperature where the Euclidean actions of both saddle points are equal

$$\beta_{\text{HP}} = 2\pi.\qquad(2.9)$$

2.2 The partition function

A comprehensive quantization of String theory on AdS_3 background was accomplished in [32–34]. In these series of beautiful papers the authors analyzed the $\mathrm{SL}(2,\mathbb{R})$ WZW model (at level k) describing strings propagating in $AdS_3 \times \mathcal{M}$, where $\mathcal{M}$ is an internal CFT which makes the theory critical. The Hilbert space of the $\mathrm{SL}(2,\mathbb{R})$ WZW model is decomposed into representation of the left/right-moving current algebra $\widehat{\mathrm{SL}(2,\mathbb{R})} \times \widehat{\mathrm{SL}(2,\mathbb{R})}$. The unitary representations contributing to the spectrum are pairs of continuous representations $\mathcal{C}^\alpha_{j=1/2+is} \times \mathcal{C}^\alpha_{j=1/2+is}$ and pairs of lowest/highest weight discrete representations $\mathcal{D}^\pm_{j>1/2} \times \mathcal{D}^\pm_{j>1/2}$. This is an oscillator expansion around geodesics with mass $m^2 = j(j-1)$. It is known that these 3 representations form a complete basis for $\mathcal{L}^2(\mathrm{AdS}_3)$ and are part of the Hilbert space of the WZW model. In addition to the conventional representation discussed above, the Hilbert space contains spectral flowed representations $\mathcal{D}^{\omega\pm}_{j>1/2} \times \mathcal{D}^{\omega\pm}_{j>1/2}$ and $\mathcal{C}^{\omega\,\alpha}_{j=1/2+is} \times \mathcal{C}^{\omega\,\alpha}_{j=1/2+is}$ with $\omega \in \mathbb{Z}$. These should be thought of as oscillator expansion around long string. They furnish representations of the spectral flowed algebra,

$$\tilde{J}^\pm_n = J^\pm_{n\pm w} \qquad\qquad \tilde{J}^3_n = J^3_n - \frac{k}{2}w\delta_{n,0}.$$

The reader is encouraged to read [32] for a complete description of the spectrum.

The $SL(2,\mathbb{R})$ theory at level k has an interesting phase structure as a function of k, all the representation discussed above exist only if $\frac{1}{2} < j < \frac{k-1}{2}$. Thus, if $k < 3$ there are no states in the Hilbert space since the vacuum[5] is projected out. A detailed study of the phase structure and its physical interpretation was conducted in [35] and references within.

The Euclidean AdS_3 background is described by the H_3^+ coset WZW model (at level k). A canonical description of the model uses the Poincaré patch coordinates for thermal AdS_3,

$$ds^2 = \frac{k}{y^2}\left(dy^2 + dw d\bar{w}\right), \tag{2.10}$$

where,

$$y = \frac{e^t}{\cosh\rho} \qquad\qquad w = \tanh\rho\, e^{t+i\theta} \qquad\qquad \bar{w} = \tanh\rho\, e^{t-i\theta}. \tag{2.11}$$

In [33] the partition function of the model was computed (following [36])

$$Z(\beta,\mu) = \frac{\beta(k-2)^{\frac{1}{2}}}{2\pi}\int_R \frac{d\tau d\bar{\tau}}{\sqrt{\tau_2}} e^{4\pi\tau_2\left(1-\frac{1}{4(k-2)}\right)}\sum_{h,\bar{h}} D(h,\bar{h})e^{2\pi i\tau(h+\bar{h})}$$
$$\times \sum_{n,m}\frac{e^{-k\beta^2|m-n\tau|^2/4\pi\tau_2 + 2\pi(Im U_{n,m})^2/\tau_2}}{|\vartheta_1(\tau,U_{n,m})|^2}, \tag{2.12}$$

where $D(h,\bar{h})$ is the degeneracy of the internal[6] CFT, τ is the modular parameter of the worldsheet torus (not to be confused with the spacetime torus used above), R is a fundamental domain of the modular group $SL(2,\mathbb{Z})$ and

$$U_{n,m}(\tau) = \frac{i}{2\pi}(\beta - i\mu\beta)(n\bar{\tau} - m). $$

The partition function has poles in the modular parameter plane at $\tau_{\text{pole}} = \frac{r}{w} + i\frac{\beta}{2\pi w}$ for integers r and w. These appear due to the long strings in the spectrum [33].

2.3 Singular conformal field theory

$AdS_3 \times S^3$ with NS-NS background is a solvable worldsheet conformal field theory. It has, therefore, been a useful laboratory to explore the AdS/CFT duality [1–3, 37]. From the spacetime point of view, however, it is a rather complicated system which does not conform to the standard ideas of an ordinary field theory on the boundary. The associated pathologies will manifest themselves in the thermal AdS/BTZ discussion as well.

The pathology that will concern us the most (other pathologies are related to it), is the fact that the conformal field theory is "singular" [38]. The singularity is similar to the $R^4/\mathbb{Z}_2$ target space with $\theta = \int B_{S^2} = 0$ where the integral is on the 2-cycle at the origin,

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[5]The vacuum is assumed to be $SL(2,\mathbb{R})$ invariant. Hence, it has vanishing Casimir $j(j-1) = 0$ which implies $j = 1$.

[6]The internal CFT has central charge such that $c_{\text{int}} + c_{SL(2,\mathbb{R})} = 26$.

and manifests itself in the spacetime CFT as a continuum of operators with continuous scaling dimension. From the spacetime point of view, these operators correspond to "long strings" - strings that stretch around θ direction in equation (2.3) at large values of ρ, which can continue to $\rho \to \infty$ if their energy is above a finite threshold. Although the proper size of the θ angle increases, and the string stretches more and more geometrically, this effect is canceled by the B field which permeates the string and the string can still make it to infinity.

The rest of the paper will be devoted to the tachyons that take the system between thermal AdS_3 ($TAdS_3$) and the $EBTZ$ black hole. Naively, and in all higher dimensions, $TAdS$ and Euclidean black holes differ by normalizable modes at the boundary of AdS, and therefore they are manifestly in the same ensemble. Correspondingly, if there is a tachyon which takes us from one configuration to the other, then it is localized in the interior of AdS and therefore is a discrete state. This is not the case for $TAdS_3$ and $EBTZ$ in the case that the background contains only NS-NS fields - the tachyons that we will encounter (for zero angular momentum) are part of a continuum, and match on to the long string spectrum at large values of ρ.[7] Since the interpolating tachyons can make it all the way to the boundary, and their geometry is different in $TAdS_3$ and $EBTZ$ (although related by $\mathrm{SL}(2, \mathbb{Z})$ of course), there is information on the boundary that distinguishes the two backgrounds. Correspondingly, $TAdS_3$ and $EBTZ$ differ by non-normalizable modes and including them in the same ensemble is done by fiat.

Peculiar as it may sound, such pathologies are not unexpected in a theory with long strings type "singularities", which are also counter intuitive from the standard UV/IR perspective.[8]

In [38] it was shown that one can go away from the locus of singular spacetime CFTs by turning on some of the moduli of the theory. In our case one goes away from the NS-NS $AdS_3 \times S^3$ background by turning on RR fluxes in the background. If the scenario that we outlined in the paragraphs above is correct, then this deformation should also change the continuum of interpolating tachyons into a discrete set. Deformations of this type have been constructed in [41, 42] and it is easy to see that this is indeed what happens.

We will refer to these background as "regulated", and discuss their application to the $TAdS_3/EBTZ$ transition in section 4.3. At this point we will just outline our approach. We will work with the NS-NS background since this is the only case that we can solve exactly, and will view the addition of R-R fields as a regulator for the behavior near the boundary of AdS_3. We are entitled to do so because in the regulated version there is no additional pathology that arises from the vicinity of the boundary.[9]

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[7]After some kinematical "processing" that will have to do with the difference between the Lorentzian space, where the "long strings" are defined, and the Euclidean space, where the interpolating tachyons are defined.

[8]Not to mention that these theories also exhibit (related) pathologies even below the long string gap such as dimension zero operator which is distinct from the identity (even on the short string sector) [39, 40].

[9]The Philosophy is similar to the one advocated in [11] where tachyons which are localized on time like singularities are discussed. The tachyons resolve and smooth the singularities but then expand outwards until, at infinite time, they will reach the boundary of space. The fact that the boundary conditions change

3. Hagedorn temperature

On general grounds we expect String theory in thermal AdS_3 to exhibit an exponential growth of the density of states which leads to the breakdown of perturbative String theory at the Hagedorn temperature. In order to compute the Hagedorn temperature we use Polchinski's trick [43] to replace the sum over string windings n in (2.12) by the sum over copies of the fundamental domain. This is interpreted as the summation over the spacetime field theory one loop amplitudes followed by a summation over the number of particles. The resulting one-loop partition function is

$$Z(\beta, \mu) = \frac{\beta(k-2)^{\frac{1}{2}}}{8\pi} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} \int_{-1/2}^{1/2} d\tau_1 e^{4\pi\tau_2\left(1-\frac{1}{4(k-2)}\right)} \sum_{h,\bar{h}} D(h,\bar{h}) e^{2\pi i\tau(h+\bar{h})}$$
$$\times \sum_{m=1}^\infty e^{-(k-2)m^2\beta^2/4\pi\tau_2} \frac{|\eta(\tau)|^4}{\left|\vartheta_{11}\left(-\frac{im\hat\beta}{2\pi}, \tau\right)\right|^2}. \quad (3.1)$$

The Hagedorn behavior of the partition function is manifest in the limit $\tau_2 \to 0$ (in field theory this is a UV limit.[10]) In this limit the partition function diverges exponentially for $\beta < \beta_H$, which corresponds to a Hagedorn transition at $T = T_H$. There is another exponential divergence for $\tau \to \infty$ due to the usual bosonic String tachyon, but it has no temperature dependence. Poles at zeros of the theta-function are interpreted in [33] as volume divergence due to worldsheet instantons which can make their way to the boundary.

The calculation of the leading order behavior in the $\tau_2 \to 0$ limit is independent of the value of τ_1 which we set to zero. We also set $\mu = 0$ which will be the focus of this paper. The contribution of the internal CFT ($\mathcal{M}$) is

$$\sum_{h,\bar{h}} D(h,\bar{h}) q^h q^{\bar{h}} \sim e^{\frac{\pi c_{\text{int}}}{6\tau_2}}, \quad (3.2)$$

combined with the contribution for the eta-functions, theta-function and exponential we find the integrand of (3.1) behaves as

$$\sim \exp\left[-\frac{k\beta^2}{4\pi\tau_2} + \frac{\pi\left(24 - \frac{6}{k-2}\right)}{6\tau_2}\right]. \quad (3.3)$$

Thus, the integrand is exponentially divergent for $\beta < \beta_H$ where,

$$\beta_H^2 = \frac{4\pi^2}{k}\left(4 - \frac{1}{k-2}\right), \quad (3.4)$$

which signals the Hagedorn behavior of the theory. A quick check of the result is to consider the flat space limit ($k \to \infty$) which produces as expected the result of [10],

$$\beta_H^{(\text{flat})} = \lim_{k\to\infty} k\beta_H^2 = 16\pi^2.$$

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between the initial worldsheet CFT and the final worldsheet CFT (at infinite distance) is irrelevant for the discussion of how the tachyons smooth the singularity.

[10] As always, this is related to an IR divergence due to a tachyonic mode by a modular transformation.

The extra factor of k is a necessary part in the double scaling limit which gives the flat-space geometry correctly.

As discussed in [10], the Hagedorn behavior can also be seen as the appearance of the Atick-Witten tachyon[11] in the action of a string wrapping the compact time coordinate as we raise the temperature (shrink the time circle). A similar calculation in thermal AdS_3, where one wraps the non contractible time circle with a winding string, reproduces the same result as the one loop partition function. We shall return to this calculation in the sequel.

The interpretation of the Hagedorn behavior in AdS space was clarified in [6–8]. Starting from low temperature the stable and dominant phase is a thermal AdS which can be viewed as a gas of strings at thermal-equilibrium. As we raise the temperature we cross the Hawking-Page temperature where thermal AdS is no longer the dominant phase but it is meta-stable. If we continue to raise the temperature forcing the system to stay in the thermal AdS state (i.e an overheated AdS) we are bound to reach the Hagedorn temperature where the thermal AdS system develops instability in perturbations theory and the system flows to another phase. For AdS_3 with vanishing chemical potential the only other dominant absolutely stable phase is the Euclidean BTZ black hole (with no angular momentum nor any RR charge), thus, as we cross the Hagedorn temperature we expect that the Thermal AdS_3 must collapse to the Euclidean BTZ black hole.

A very interesting property, unique to AdS_3, is the diffeomorphism between $TAdS_3$ with identification τ and $EBTZ$ with identification $\frac{-1}{\tau}$, setting $\mu = 0$ the diffeomorphism is between $TAdS_3$ at temperature T and $EBTZ$ at temperature $\frac{1}{4\pi^2 T}$. Therefore we can move backwards on the phase diagram just described - starting with $EBTZ$ at high temperature cooling the system through the Hawking-Page temperature to a overcooled (meta-stable) $EBTZ$ up to the point we reach yet another Hagedorn temperature where the system must collapse to $TAdS_3$. The complete phase diagram is sketched in figure 1.

From the $EBTZ$ point of view there is also an analog of the tachyon winding the time circle in $TAdS_3$. Using the diffeomorphism between $EBTZ$ and $TAdS_3$ we learn that when we lower the $EBTZ$ temperature a string winding the angular coordinate develops a tachyonic mode exactly where the diffeomorphic $TAdS_3$ develops the Atick-Witten tachyon,

$$T_c^{(EBTZ)} = \frac{1}{2\pi\sqrt{k}} \left(4 - \frac{1}{k-2} \right)^{1/2} = \frac{4\pi^2}{T_H^{(TAdS)}}. \tag{3.5}$$

Note that the proper size of the angular circle of the $EBTZ$ black hole when one reaches this temperature is of the order of the String scale. In spite of the fact we have obtained this result from Euclidean considerations, it has an interesting Lorentzian interpretation as well. One can continue the $EBTZ$ black hole back to Lorentzian signature by rotating the time circle. Near the point where the mode becomes tachyonic, the resulting black hole is small with horizon area of the order of the String scale. Thus, we have calculated exactly the point where the tachyon which is expected to be generically behind the horizon of the

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[11]The best definition for a tachyon in Euclidean theory is an exponential divergence in the partition function related to IR physics in space time.
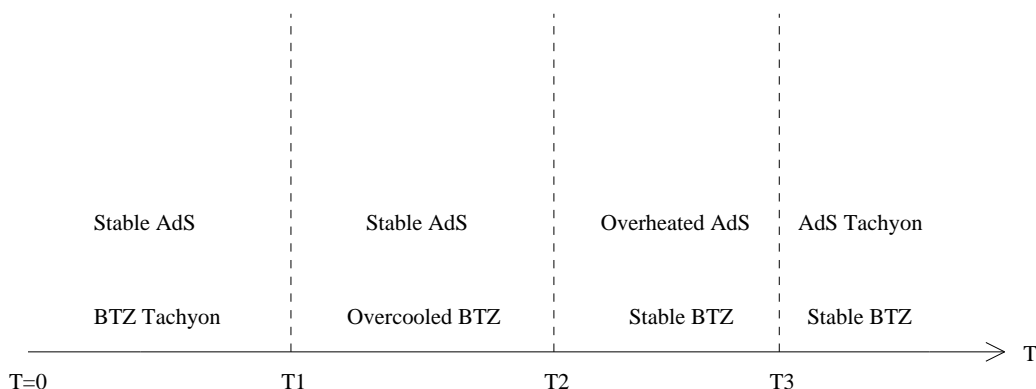
Figure 1: Phase diagram for the asymptotic AdS_3 boundary conditions with chemical potential $\mu = 0$. As we explained, the $EBTZ$ tachyon is winding around the angle of the $EBTZ$ while the $TAdS_3$ tachyon winds around the compact time of the $TAdS_3$. The various temperatures are: $T1 = \frac{1}{2\pi\sqrt{k}}\left(4 - \frac{1}{k-2}\right)^{1/2}$, $T2 = \frac{1}{2\pi}$, $T3 = \frac{\sqrt{k}}{2\pi}\left(4 - \frac{1}{k-2}\right)^{-1/2}$.

BTZ black hole[12] "leaks" outside the horizon. This is closely related to the phenomenon discussed (mainly in the AdS_5 context) in [9].

The above result is exact in α' (equivalently in k) enabling to probe the theory in the strongly curved limit. An immediate feature of the above formula, is the behavior for $k = 3$ where,

$$k \to 3: \qquad T_c^{(EBTZ)} = T_H^{(TAdS)} = T_{\mathrm{HP}} = 2\pi. \qquad (3.6)$$

Consequently, the region $k < 3$ does not make any sense physically[13] because the theory develops an instability before one reaches the phase transition point. In fact, as explained in [35], when $k = 3$ the Lorentzian $\mathrm{SL}(2,\mathbb{R})$ model loses its vacuum. Therefore the interpretation of the model as string propagating in AdS_3 is no longer valid. It will be nice to understand better the relation between these two pathologies, we further discuss this in the next subsection.

One can analyze the properties of the Atick-Witten tachyon[14] by a minisuperspace calculation of a string winding around the time circle. The details of the calculation are somewhat technical so we leave it to appendix A. The main result of the calculation is the eigenvalue equation describing the mode (A.6),

$$E\Psi(\rho) = \frac{1}{2k}\left[-\partial_\rho^2 - 2\coth(2\rho)\partial_\rho + \left(\frac{\beta k}{2\pi}\right)^2 - a\right]\Psi(\rho), \qquad (3.7)$$

where a is related to the zero-point energy constant. It is calculated, by employing canonical quantization, in appendix A as well. The winding string moves in a potential which is

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[12]Due to the orbifold singularity behind the horizon.

[13]Naively, one expects the theory to break down at $k = 2$ where the energy momentum tensor ceases to exist but not before.

[14]We thank S. Minwalla for very instructive discussions on this point, and in particular on the delocalization of the tachyon.

bounded at infinity. While naively from the metric one expects an infinitely high potential as $\rho \to \infty$, the presence of the B-field in the background cancels the $e^{2\rho}$ term coming from the geometry. At large ρ the wave function of the winding mode has the following dependance (see (A.7))

$$\Psi_j(\rho) \sim e^{+2j\rho} \qquad j = -\frac{1}{2} + is \qquad E_j = \frac{1}{2k}\left(-4j(j+1) + \frac{\beta^2 k^2}{4\pi^2} + a\right), \qquad (3.8)$$

where s is a real parameter. After fixing the constant a, one discovers that at the Hagedorn temperature $\beta = \beta_H$ the lowest mode ($s = 0$) has exactly zero energy $E_{-1/2} = 0$. It is important to note that the winding mode is delta-function normalizable.[15] This indicates that the condensation of this mode is not a strictly local process. As we discussed in section 2.3 this oddity is related to the fact that the spacetime CFT is singular, and it is removed when the background is appropriately regulated.

3.1 More on $k = 3$

As argued above, the ensemble does not have sensible thermodynamic properties below $k = 3$. This fits well with the fact that the Lorentzian vacuum of SL(2, $\mathbb{R}$) does not exist below $k = 3$, but one can be more precise. A host of special effects which take place at $k = 3$ were discussed in [35].[16] In addition to projecting out the vacuum, it was found that the BTZ black hole becomes non normalizable and that the correct description below $k = 3$ is by weakly coupled long strings (one can also check that the gap for long strings disappears when $k = 3$).

This culmination of observations makes the connection to our result more obvious.[17] For $k > 3$, the thermodynamic description is the standard one, as depicted in figure 21. A generic high energy state is the black hole, and the gas of particles ceases to exist beyond the Hagedorn temperature. When $k = 3$, the states smoothly go over to each other, and it is manifest that the Hagedorn and Hawking-Page temperatures match. This situation is depicted in figure 2b. Lastly, the only thing that can happen below $k = 3$ and be consistent with everything said above is that the BTZ black hole can never be reached (the system has a physical limiting temperature), and a generic state is just a weakly coupled long string. This is displayed in figure 2c where the line of putative black holes, above a gap in temperature, is also exhibited (of course, it can never be reached) for making it easy to visualize how the transition[18] at $k = 3$ occurs.

4. The competing winding modes picture

$TAdS_3$ and $EBTZ$ are different ways of extending the $S^1 \times S^1$ boundary to a 3 dimensional manifold. In the two cases, one fills different S^1's into a disk D_2. When one of
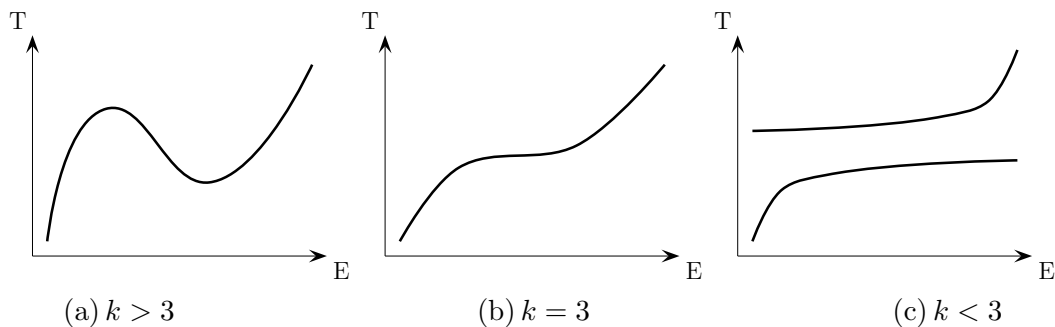
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[15]In the Klein-Gordon norm there is a volume factor $\sim \sinh(2\rho)$ that cancels the exponential decay of the wave function.

[16]These effects are associated to the black hole/black string transition which occurs exactly at $k = 3$.

[17]We are grateful to O. Aharony for discussions on this point.

[18]From a system that has no limiting temperature to one that has.

(a) $k > 3$ (b) $k = 3$ (c) $k < 3$

Figure 2: The vertical axis is the temperature of the system and the horizontal axis is the typical energy of an excitation. We exhibit the transition in the form of the phase diagram from $k > 3$ to $k = 3$ and finally to $k < 3$. The most dramatic change is the appearance of a limiting temperature when $k < 3$, which is closely related to the fact that typical high energy excitations are not black holes anymore but long fundamental strings.

the backgrounds becomes unstable, an Atick-Witten tachyon appears which is a winding mode on the non-contractible circle. Hence, the flow dynamics is controlled by a competition between a geometric capping, by D_2, and a Stringy capping, by a winding tachyon. Fortunately, as we will discuss in this section, in String theory these two ways of capping are indistinguishable (in a very precise technical sense). This becomes apparent in a dual formulation which contains a sine-Liouville (s-L) theory, and is the main motivation for investigating this relation.

The picture we will end up with consists of two deformations of linear dilaton by s-L walls, whose profile grows as we go into the bulk of space. They both tend to shrink their respective circles. The one with the larger coefficient on the worldsheet "wins" and caps its circle first. The problem of quantifying the phase transition is now just a problem of comparing the coefficients of two operators on the worldsheet. Further, we show that this perturbation of s-L becomes the Atick-Witten tachyon vertex operator exactly at the Hagedorn temperature. Consequently, the issues of geometric capping and tachyon condensation are closely related in our backgrounds.

This approach also provides information about the CFT that corresponds to the unstable fixed point "between" the $EBTZ$ and $TAdS_3$ (the sense in which this is "between" the two phases will be specified precisely). This is discussed in section 5.

Intuitively, the idea is as follows. Thermal AdS_3, in which the disk fills up the angular direction, is represented in figure 3a. Suppose the temperature is high enough so that a tachyon develops on the non contractible circle. On general grounds, one would expect the thermal circle to pinch and the space to cut and begin from some larger value of the radial coordinate $\rho \geq \rho_0$. The resulting space is easy to visualize. It has a contractible thermal circle and a non contractible angular variable. This is precisely what we need for a Euclidean black hole. Indeed, the Euclidean BTZ black hole (whose topological structure is exhibited in figure 3b) is our candidate for the end point of tachyon condensation.

The picture is reversed when we consider a very cold Euclidean BTZ black hole-

(a) Thermal AdS_3 (b) Euclidean BTZ
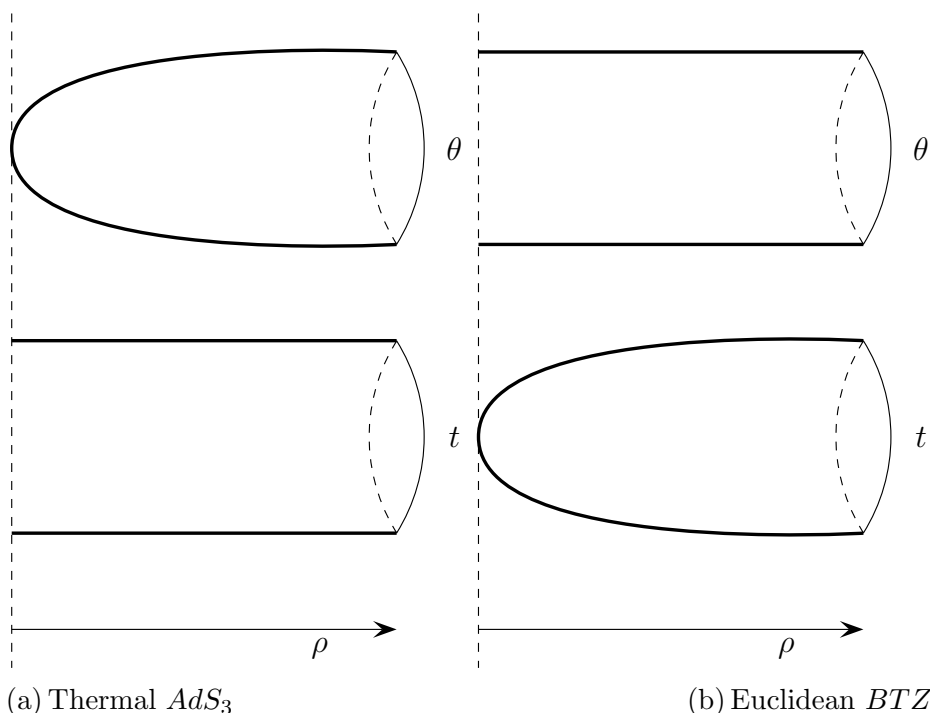
Figure 3: The topological structure of thermal AdS_3 on the left and Euclidean BTZ black hole on the right. The non contractible circles are the thermal (time) circle for $TAdS_3$ and the angular (spatial) circle in $EBTZ$.

the disk fills the Euclidean time circle and the tachyon develops along the angular circle contracting it. Therefore, tachyon condensation switches the topology to that of a $TAdS_3$.

However, we will see that the dynamics is not so simple. As in the $\mathbb{C}/\mathbb{Z}_N$ case considered in [11], there is a need for an outgoing shell to change some boundary conditions at infinity (after infinite RG flow time). Otherwise, the flow can not end in a smooth background. The physics we find here is very similar in some respects.

For convenience we outline the procedure we employ to derive our dual description

- Begin with either $TAdS_3$ of $EBTZ$ (the end result will be the same).

- By twice T dualizing non contractible circles, we write the background as $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \mathrm{U}(1)$. This is done in section 4.2.

- Using FZZ duality (reviewed in section 4.1), the latter is a linear dilaton theory perturbed by a winding mode condensate, a "wall", in the bulk. The geometric cap is replaced by a winding mode cap.

- The model is now a linear dilaton with two circles and a winding mode condensate on one of them. If one is at the Hagedorn temperature, turning on the Atick-Witten tachyon on the Euclidean time direction amounts to turning on another sine-Liouville

interaction on the other circle. The picture is now completely symmetric between the spatial and time circles where there are two vertex operators of the same kind.

In the range of temperatures in which both $TAdS_3$ and $EBTZ$ are perturbatively stable, the thermodynamic description (see figure 21) suggests that there is an unstable fixed point between these two stable fixed points. One can guess how to construct this unstable point.[19] What we have basically done is to describe two distinct ways to cap, in a non singular way, the model $linear\ dilaton \times S^1 \times S^1$. In $TAdS_3$ we have filled with a disc one of the circles while in $EBTZ$ we have filled with a disc the other. To obtain the intermediate point one needs to treat the two circle more symmetrically. One can either

- Not cap either of the circles. In this case the model is simply $linear - dilaton \times \mathbb{T}^2$. This point does not exist in the space of perturbative String theory. It may exist once non-perturbative effects are taken into account (similar to the distinction between LST and DLST backgrounds [44 − 47]).

- Cap both circles at the same time. If one can truncate the dynamics to the coefficient of the two sine-Liouville interaction then such a point must exist simply because there must exist an unstable fixed point between two stable fixed point if there in one real parameter.[20] This capping is perturbative in g_s.

- Some other way of capping, such as a Liouville cap (without winding condensation).

We return to the question of constructing the unstable phase in section 5. In section 4.3, we discuss a way to regulate the singularities reviewed in section 2.3 and explain the physical consequences of such a regulator.

4.1 A review of the FZZ duality

The FZZ duality, first conjectured by V.Fateev, A.Zamolodchikov and Al.Zamolodchikov [24], is a correspondence between the $\text{SL}(2)_k/\text{U}(1)$ coset CFT and sine-Liouville theory (see [48 − 51] and references therein for relevant works on the subject). We shortly review the duality following [48].

The $\text{SL}(2)_k/\text{U}(1)$ coset CFT [52 − 54] also known as the 2-dim Euclidean black hole or cigar model, describes the geometry

$$ds^2 = k \left(dr^2 + \tanh^2 r\, d\theta^2 \right) \qquad\qquad \Phi - \Phi_0 = -2 \log \cosh r, \qquad (4.1)$$

where θ is a periodic coordinate $\theta \cong \theta + 2\pi$ and r ranges from 0 to ∞. At the boundary of space, $r \to \infty$, the model reduces to a product of a linear dilaton and a circle, and at the tip of the cigar, $r = 0$, the circle pinches. The geometry is smooth everywhere, and the String coupling $g_s = e^{\Phi_0}$ at the tip is finite. The level k is a free parameter which governs the size of the cigar. The mass of the underlying Lorentzian black hole is related to the String coupling at the tip of the cigar $M \propto e^{-2\Phi_0}$.

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[19]Of course, the fact it is unstable at intermediate temperatures implies that it should be, in some sense, a singular solution. This is analogous to the small black hole in the AdS_5 ensemble.

[20]We will see that the situation is slightly more complicated than that.

The central charge of the cigar CFT (4.1) is,

$$c = \frac{3k}{k-2} - 1. \tag{4.2}$$

For the special case $k = \frac{9}{4}$ the solution is a two dimension bosonic String theory dual to a $c = 1$ matrix model. For larger values of k one needs to multiply the model by a suitable CFT to complete the central charge to 26. An important set of observables corresponds to momentum and winding modes on the cigar $V_{j,m\bar{m}}$. These have scaling dimensions

$$\Delta_{j,m,\bar{m}} = -\frac{j(j+1)}{k-2} + \frac{m^2}{k} \qquad\qquad \bar{\Delta}_{j,m,\bar{m}} = -\frac{j(j+1)}{k-2} + \frac{\bar{m}^2}{k}. \tag{4.3}$$

The parameters $m, \bar{m}$ are related to the winding and momentum quantum numbers around the θ circle at large r,

$$m = \frac{n + wk}{2} \qquad\qquad \bar{m} = -\frac{n - wk}{2} \qquad\qquad n, w \in \mathbb{Z}. \tag{4.4}$$

The geometry of the cigar, which is topologically a plane, makes it obvious that the cigar background conserves only the momentum $m - \bar{m}$ and not the winding $m + \bar{m}$,

String perturbation theory (genus expansion) is an expansion in $e^{2\Phi}$, which can be thought as an expansion in the String coupling at the tip $e^{2\Phi_0} \sim \frac{1}{M}$. The geometry has radius of curvature k, and thus the worldsheet is weakly coupled for large k. However the coset model is well defined for small values of k as well.[21]

The dual sine-Liouville theory is defined by the 2-dim Lagrangian density

$$4\pi\mathcal{L} = (\partial\phi)^2 + (\partial x)^2 + Q\hat{R}\phi + \lambda e^{b\phi} \cos R\,(x_L - x_R)\,, \tag{4.5}$$

where the x direction is periodic with

$$x \cong x + 2\pi R \qquad\qquad R = \sqrt{k}, \tag{4.6}$$

and the theory has a linear dilaton $\Phi - \Phi_0 = -\frac{Q}{2}\phi$. The central charge of this theory is $c = 2 + 6Q^2$. Comparing with (4.2) we set,

$$Q^2 = \frac{1}{k-2}. \tag{4.7}$$

The s-L interaction is the lowest lying winding mode (winding number equals one) around the circle x, the exponent b is fixed by requiring that the interaction is marginal,

$$\frac{1}{4}R^2 - \frac{1}{4}b(b + 2Q) = 1 \quad \Rightarrow \quad b = -\frac{1}{Q} = -\sqrt{k-2}. \tag{4.8}$$

There are two inequivalent models: $\lambda \neq 0$ and $\lambda = 0$. In the first case, the coefficient $\lambda > 0$ can be rescaled to any other positive value by a shift in ϕ (and re-absorbed in the String coupling). If we set λ to zero we are left with the linear dilaton theory (with the

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[21]There are some $\frac{1}{k}$ corrections to the action, as explained in [55].

coupling diverging at $\phi \to -\infty$) which is ill defined in perturbation theory. In [48] the authors treated the s-L theory as a perturbation of Liouville theory, i.e adding a Liouville potential $\mu_L \phi\, e^{-2\phi}$ and treating λ as a perturbation. Varying the radius R of the x circle they found that at a critical of radius $R = \sqrt{k}$ one can take the limit $\mu_L \to 0$ with no singularities.[22] Naively, the s-L theory has an infinite coupling region ($\phi \to -\infty$), but the interaction terms generates a wall preventing particles from reaching the strong coupling area (similar to the cosmological constant potential $\mu_L \phi e^{-2\phi}$ in the more familiar Liouville theory).

Wave functions of s-L theory behave at large ϕ as

$$\Psi(\phi) \sim e^{\left(Q - \frac{1}{Q}\right)\phi}. \tag{4.9}$$

At large k the wave function Ψ goes rapidly to zero at the weak coupling area ($\phi \to \infty$), thus, the theory is effectively strongly coupled. On the other hand if $k \to 2$ ($Q \to \infty$) the wave function Ψ is supported at large ϕ away from the potential wall. Consequently, s-L theory is strongly coupled when the cigar CFT becomes weakly curved and vice-versa.

The observables $V_{j,m,\bar{m}}$ of the cigar are mapped to vertex operators of s-L which have the following large ϕ behavior (for large k this is simply the requirement that the asymptotics of the vertex is the same):

$$V_{j,m,\bar{m}} \leftrightarrow e^{ip_L x_L + ip_R x_R + \beta\phi}, \tag{4.10}$$

with

$$p_L = \frac{n}{R} + wR \qquad\qquad p_R = \frac{n}{R} - wR \qquad\qquad \beta = 2Qj.$$

At large ϕ and r ($\phi, r \to \infty$) both s-L theory and the cigar model describe a cylinder with linear dilaton implying the identification (at large k)

$$r \sim -Q\phi \qquad\qquad\qquad \theta \sim \frac{x}{\sqrt{k}}. \tag{4.11}$$

FZZ duality is the statement that the cigar coset model and s-L theory are exactly equivalent as conformal field theories. As explained above, this is interpreted as a strong-weak duality of the worldsheet theories.

S-L correlation functions have a KPZ behavior [56] implying that the partition sum has the following genus expansion,

$$\mathcal{F}(\lambda, g_s) = \sum_{h=0}^{\infty} \mathcal{F}_h \left(g_s \lambda^{-\frac{1}{k-2}}\right)^{2(h-1)}. \tag{4.12}$$

Thus, $g_s \lambda^{-\frac{1}{k-2}}$ is an effective String coupling and one can set $g_s = 1$ for convenience. On the other hand, genus expansion of String theory on the cigar is related to the String

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[22]The calculation in [48] was done only for $k = 9/4$ due to the technical need of using matrix model results. The fact there is an effective wall due to the s-L interaction is well established for any k by investigating three point functions (for more details see [49] and references therein).

coupling at the tip of the cigar $g_s^2 \sim 1/M$ (where M is measured in planck units). Therefore in the FZZ duality the correspondence of the genus expansions suggests,

$$M \leftrightarrow \lambda^{\frac{2}{k-2}}. \tag{4.13}$$

There is no complete proof of the bosonic conjecture (see [57] for recent review), but strong evidence comes from comparison of 2-pt and 3-pt functions. The supersymmetric version of the duality is given in [46], Hori and Kapustin [58] used gauged linear sigma-model techniques to show that in this case the duality is an example of mirror symmetry. For recent discussion on the supersymmetric duality and it's relation to the bosonic conjecture see [59 – 61]

4.2 $TAdS_3$ and $EBTZ$ using FZZ duality

4.2.1 Step 1: The background as $cigar \times S^1$

Recall the solution of the thermal AdS_3 background[23]

$$\begin{aligned} ds^2 &= k \left(d\rho^2 + \cosh^2 \rho \, dt^2 + \sinh^2 \rho \, d\theta^2 \right) + ds_\perp^2 \\ B_{(2)} &= -2ik \sinh^2 \rho \, dt \wedge d\theta \\ e^\Phi &= g_s, \end{aligned} \tag{4.14}$$

with the identifications

$$\theta \cong \theta + 2\pi \qquad , \qquad t \cong t + \beta. \tag{4.15}$$

The B-field is imaginary since it is an analytic continuation of a real B-field in Lorentzian space. We T-dualize [62] the non contractible time circle to obtain

$$\begin{aligned} ds^2 &= k \left[d\rho^2 + \tanh^2 \rho \, (d\theta - i \, d\tilde{t})^2 + d\tilde{t}^2 \right] + ds_\perp^2 \\ B_{(2)} &= 0 \\ \Phi' &= \log(g_s) - \log\left(\cosh^2 \rho \right) - \log\left(\frac{k\beta^2}{4\pi^2} \right), \end{aligned} \tag{4.16}$$

with the identifications,

$$\theta \cong \theta + 2\pi \qquad , \qquad \tilde{t} \cong \tilde{t} + \frac{4\pi^2}{k\beta}. \tag{4.17}$$

Note that some key features in the geometry at infinity have apparently changed. In (4.14), the boundary is only conformal to $\mathbb{T}^2$ with a scale factor that diverges at infinity. In (4.16) the metric of the boundary, in the String frame, is a finite $\mathbb{T}^2$. The interpretation of this result is that in the original picture, (4.14), the energy of winding strings that go to the boundary is a finite constant (this is derived in a mini-superspace framework in appendix A).

The background in the ρ-θ plane is the familiar Euclidean cigar background (also known as the 2-dim Euclidean black hole) studied in [52 – 54], if we define $d\chi = d\theta - i \, d\tilde{t}$

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[23]We use the wedge product normalization $\quad a \wedge b = \frac{1}{2}(a \otimes b + b \otimes a)$.

and treat it as if it has the usual reality condition of a real scalar.[24] We can then use known results about the cigar. For example, as a quick check, one can compare the central charge of the $cigar \times S^1$ background with the central charge of $TAdS_3$,

$$c_{(AdS_3)} = \frac{3k}{k-2} = 3 + \frac{6}{k-2} \qquad c_{(cigar \times S^1)} = \left(2 + \frac{6}{k-2}\right) + 1.$$

This is, of course, not surprising.

Next, we apply another T-duality in the decoupled $\tilde{t}$ circle direction (such that the $d\chi$ direction not effected by the transformation), we find the following background (denoting the T-dual coordinate to $\tilde{t}$ by φ)

$$\begin{aligned} ds^2 &= k \left(d\rho^2 + \tanh^2 \rho \, d\chi^2 + d\varphi^2\right) + ds_\perp^2 \\ B_{(2)} &= 0 \\ \Phi' &= \log(g_s) - \log\left(\cosh^2 \rho\right) \quad \Rightarrow \quad g_s' = g_s \, / \, \cosh^2 \rho, \end{aligned} \qquad (4.18)$$

with the identifications,

$$\chi \cong \chi + 2\pi \qquad , \qquad \varphi \cong \varphi + \beta. \qquad (4.19)$$

Our result can be summarized as follows

$$TAdS_3 = \left(\frac{\mathrm{SL}(2,\mathbb{C})_k}{\mathrm{SU}(2)} \, \Big/ \, \mathrm{U}(1)\right) \times \mathrm{U}(1)_{\beta, \, k}, \qquad (4.20)$$

where the notation $\mathrm{U}(1)_{\beta, \, k}$ is there to remind us that the inverse temperature is β but the proper size of the decoupled circle is $\beta\sqrt{k}$. This decomposition is similar to the one suggested in [63] for the Lorentzian signature backgrounds. One immediate surprise is encountered if one tries to repeat this exercise in the case of the $EBTZ$ black hole. The easiest way to do it for the black hole is to exploit its equivalence to a Thermal AdS_3 with a different temperature,

$$EBTZ = \left(\frac{\mathrm{SL}(2,\mathbb{C})_k}{\mathrm{SU}(2)} \, \Big/ \, U(1)\right) \times \mathrm{U}(1)_{4\pi^2/\beta, \, k}. \qquad (4.21)$$

In both (4.20) and (4.21), the asymptotic form of the metric is just $linear - dilaton \times S^1 \times S^1$. For both $TAdS_3$ and $EBTZ$ one pinches the circle of proper size $2\pi\sqrt{k}$ and the other remains intact. *However*, the decoupled circles have different proper size at infinity. This is surprising at first sight, since one expected $TAdS_3$ and $EBTZ$ to be in the same thermal ensemble (when they have the same temperature). Indeed, they are in the same ensemble when one does pure gravity. String theory, on the other hand, clearly has a non normalizable mode distinguishing the two backgrounds. This is disguised in the original description but it is manifest in this T dual frame, where this non normalizable mode is a proper size of a free circle at infinity.

Thus, it appears that $TAdS_3$ with modular parameter τ and $TAdS_3$ with modular parameter $\frac{-1}{\tau}$ are not in the same thermal ensemble in the framework of String theory.

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[24]One can write down the dictionary for vertex operators, taking this factor i into account. However, we will not need to deal with it for a reason which will soon become clear.

One would like to have an understanding of this pathology in the original frame as well. The naive way to characterize this is to recall that long strings (in the original frame) have constant energy potential near the boundary. The value of this constant is different in the two backgrounds as can be seen from the explicit computations in appendix A. There is a more elegant description in the original frame which is explained in the sequel.

Consider the asymptotic form of $TAdS_3$, keeping the first sub-leading correction to the B field

$$ds^2 \simeq k \left(d\rho^2 + \frac{e^{2\rho}}{4} dt^2 + \frac{e^{2\rho}}{4} d\theta^2 \right)$$

$$B_{(2)} = -2ik \left(\frac{e^{2\rho}}{4} - \Lambda_{(TAdS)} \right) dt \wedge d\theta. \tag{4.22}$$

where $\Lambda_{(TAdS)} = \frac{1}{2}$. The identifications of $TAdS_3$ with modular parameters τ and $\frac{-1}{\tau}$ are respectively

$$TAdS_3: \quad t \cong t + \beta \qquad \theta \cong \theta + 2\pi$$

$$EBTZ: \quad t \cong t + \frac{4\pi^2}{\beta} \qquad \theta \cong \theta + 2\pi.$$

We can bring the two conformal $\mathbb{T}^2$'s to have the same periodicities by a shift in ρ, however this shift changes the constant in the B-field. For concreteness shifting ρ in $EBTZ$ results in

$$\rho \to \rho + \log \frac{2\pi}{\beta} \quad \Rightarrow \quad \Lambda_{(EBTZ)} - \Lambda_{(TAdS)} = \frac{1}{2} \left(\frac{\beta^2}{4\pi^2} - 1 \right). \tag{4.23}$$

One could think that this constant mode of the B field is a an unphysical pure gauge mode. This is not true in the thermal case since the integral

$$\int_{\mathbb{T}^2} B$$

plays the role of a generalized Wilson line, which has a physical effect on the spectrum of strings. Thus, the calculation above suggests that the invariant way to characterize the non normalizable mode which distinguishes the two backgrounds is indeed $\int_{\mathbb{T}^2} B$. This fits well with the previous explanation involving the potential energy of long strings, since it is exactly this Wilson line which sets this constant.[25]

For completeness, it remains to show explicitly that this constant translates directly to the volume of the two torus in the T dual frame. To demonstrate this, we begin with the $TAdS_3$ metric with the undetermined constat Λ in the B-field

$$B'_{(2)} = -2ik(\sinh^2 \rho + \delta\Lambda) \, dt \wedge d\theta. \tag{4.24}$$

As before, we first T dualize the non contractible Euclidean time direction and get the following metric (with $B = 0$)

$$ds^2 = k \left[d\rho^2 + \frac{\left(d\tilde{t} - i\delta\Lambda \, d\theta \right)^2}{\cosh^2 \rho} - 2i \tanh^2 \rho \, d\tilde{t} d\theta + \tanh^2 \rho (1 - 2\delta\Lambda) \, d\theta^2 \right] + ds_\perp^2. \tag{4.25}$$

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[25]Note that this mode exists only near the boundary of space, since in the full background, of the circles of the two torus in contractible.

We define new complex coordinates

$$\chi = \sqrt{1 - 2\delta\Lambda}\ \theta - i\frac{\tilde{t}}{\sqrt{1 - 2\delta\Lambda}} \qquad\qquad \tilde{t}' = \frac{\tilde{t}}{\sqrt{1 - 2\delta\Lambda}}.$$

Repeating our procedure we apply a second T-duality on the new $\tilde{t}'$ to find the a deformed cigar background

$$
\begin{aligned}
ds^2 &= k\left[d\rho^2 + \frac{1 - 2\delta\Lambda}{(1 - \delta\Lambda)^2 - \delta\Lambda^2 \tanh^2 \rho}\left(\tanh^2 \rho\, d\chi^2 + d\varphi^2\right)\right] + ds_\perp^2 \\
B_{(2)} &= -2ik\frac{(1 - \delta\Lambda)\delta\Lambda}{(1 - \delta\Lambda)^2 \cosh^2 \rho - \delta\Lambda^2 \sinh^2 \rho}\, d\varphi \wedge d\chi \\
\Phi' &= \log(g_s) - \log\left(\cosh^2 \rho\right) - \log\left(1 + \frac{\delta\Lambda^2}{(1 - 2\delta\Lambda)\cosh^2 \rho}\right) + \log(1 - 2\delta\Lambda) \\
\chi &\cong \chi + 2\pi\sqrt{1 - 2\delta\Lambda}\,, \\
\varphi &\cong \varphi + \beta\sqrt{1 - 2\delta\Lambda}.
\end{aligned}
\tag{4.26}
$$

A fast consistency check is to verify that for $\delta\Lambda = 0$ the above result is the same as (4.18), i.e the usual cigar metric.[26]

To prove our assertion of the relation between the B field in the AdS_3 picture and the asymptotic volume in the T dual frame we analyze the asymptotic form of (4.26), the B-field vanishes and the asymptotic behavior of the metric is that of a $\mathbb{T}^2$ with radii

$$R_\chi = \sqrt{k}\,\sqrt{1 - 2\delta\Lambda} \qquad\qquad R_\varphi = \sqrt{k}\,\frac{\beta}{2\pi}\,\sqrt{1 - 2\delta\Lambda},$$

which is the original torus rescaled by $\sqrt{1 - 2\delta\Lambda}$, as claimed.

Consequently, the Euclidean BTZ black hole and $TAdS_3$ have different asymptotic values of the generalized Wilson lines, which inevitably leads to a non normalizable mode distinguishing them. This prevents any possible RG flow which takes finite RG time from the usual black hole and $TAdS_3$. However, there can still exist flows which emit a propagating shell affecting some boundary conditions as in [11]. Note that if we are exactly at the Hawking-Page temperature, then this difference does not exist.

The implication of our result to Lorentzian physics requires clarification. The integral $\int_{\mathbb{T}^2} B$ has no analogue in the Lorentzian case since the time coordinate is non compact. In particular, there is no obstruction for creating in a hot Lorentzian AdS_3 an excitation which is the BTZ black hole. The correct interpretation of our result is that the thermal ensemble of this system is somewhat pathological, and should be considered with care. The micro-canonical ensemble, on the other hand, follows the standard expectations.

This brings us to the need of establishing deformations of the model which have sensible canonical thermodynamic descriptions. It is in these cases where everything one

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[26]The metrics we obtain with such T dualities are always not real, but can be made real by analytically continuing the Euclidean time. In particular, the background above, if continued to Lorentzian signature is a perfectly well defined solution to the equations of motion, and is worth understanding better. It is interesting that one can generate non trivial solutions in the T dual frame by dialing a seemingly trivial mode in the original frame.

expects from this system is satisfied without subtleties. We discuss such a deformation in section 4.3.

4.2.2 Step 2: FZZ duality on the cigar

So far we have written the background as $cigar \times S^1$. We still have two kinds of caps - one is geometric (the cigar pinching) and the other via a condensing winding mode on the separate S^1. In String theory, however, these two kinds of caps are the same. The cleanest example is the FZZ duality (which we reviewed before).

Applying the FZZ duality to the cigar metric, we find the s-L lagrangian (with additional circle),

$$4\pi\mathcal{L} = (\partial\phi)^2 + (\partial x)^2 + (\partial\varphi)^2 + Q\hat{R}\phi + \lambda e^{b\phi}\cos R_x\left(x_L - x_R\right). \tag{4.27}$$

The central charge of this theory is $c = 3 + 6Q^2$, the dilaton slope is

$$\Phi - \Phi_0 = -\frac{Q}{2}\phi \qquad\qquad Q^2 = \frac{1}{k-2}. \tag{4.28}$$

The radii of the circles x and φ are

$$x \cong x + 2\pi R_x \qquad\qquad R_x = \sqrt{k}$$
$$\varphi \cong \varphi + 2\pi R_\varphi \qquad\qquad R_\varphi = \frac{\beta\sqrt{k}}{2\pi}\,, \tag{4.29}$$

and the exponent b calculated in (4.8). We adopt the view of [48] defining the s-L theory as the $\mu_L \to 0$ limit of a Liouville theory with a s-L interaction.

The s-L interaction term in (4.27) is clearly a winding mode on the x circle. Tracing back the FZZ and T dualities, the winding around the non contractible time circle (which the Atick-Witten tachyon of $TAdS_3$ winds around above the Hagedorn temperature) is related to winding around the φ circle in the s-L theory.[27]

The picture now is more symmetric - both circles can be capped with winding mode condensation. Roughly, one can consider the more general deformation of the two circles by

$$\mathcal{L} = \mathcal{L}_0 + \lambda_1 e^{b_1\phi}\cos R_x(x_L - x_R) + \lambda_2 e^{b_2\phi}\cos R_\varphi(\varphi_L - \varphi_R), \tag{4.30}$$

where $\mathcal{L}_0$ is the $linear-dilaton \times \mathbb{T}^2$ lagrangian, and the values of b_1 and b_2 are determined by the marginality requirement (at tree level)

$$\frac{1}{4}R_x^2 - \frac{1}{4}b_1(b_1 + 2Q) = 1 \qquad\qquad \frac{1}{4}R_\varphi^2 - \frac{1}{4}b_2(b_2 + 2Q) = 1\,. \tag{4.31}$$

We can shift ϕ to change both coefficients λ_1 and λ_2 and only the ratio $\eta = \lambda_1^{1/b_1}/\lambda_2^{1/b_2}$ has physical significance. Allowing the two deformation theory (4.30) to flow to its IR fixed point we expect that, for generic value of η, one of the deformations dominates over the other, driving the other to zero (and shrinking the S^1 around which the dominant

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[27]The mapping of these states is exact and does not involve subtleties in the mapping of the corresponding vertex operators.

perturbation winds). Simply, the deformation whose associated wall is closer to the weakly coupled boundary is expected to dominate.

The analysis whether such a flow generates propagating waves which can, after infinite RG time flow, change the boundary conditions (namely the proper size of the circles) is left for future work. It would also be interesting to study (4.30) using perturbations around Liouville but we leave this for future work as well. A more careful account of the possible flows is postponed to section 5.

A special point which does not suffer from the difficulties of changing the boundary conditions is the Hawking Page point where all the circles are of the same size. The physical parameter is $\eta_s = \lambda_1/\lambda_2$, and the theory has enhanced symmetry at $\eta_s = 0, 1, \infty$. We will explore this point, and some other structures associated with constructing the unstable phase, in section 5.

It is interesting to see how the basic physical features of AdS_3 are actually encoded in s-L theory. For instance, we would like to see how the Atick-Witten tachyon appears in the s-L background at exactly the correct temperature. We describe the model perturbatively in λ_2 around a $sine - Liouville \times S^1$ (λ_1 held fixed)

$$\Delta\mathcal{L} = \lambda_2\mathcal{V} = \lambda_2 V_{j,m,\bar{m}} \cos R_\varphi(\varphi_L - \varphi_R), \tag{4.32}$$

where $V_{j,m,\bar{m}}$ are the vertex operator of the 2-dim s-L theory (the $x - \phi$ plane) defined in section 4.1. The asymptotic behavior (large ϕ) of the operator is,

$$V_{j,m,\bar{m}} \xrightarrow{\phi \to +\infty} \frac{e^{i\sqrt{\frac{2}{k}}(m\chi_L - \bar{m}\bar{\chi}_R)}}{1 + 2j} \left[e^{2Qj\phi} + R_{j,m,\bar{m}} e^{-2Q(j+1)\phi} + \cdots \right]. \tag{4.33}$$

The coefficient $R_{j,m,\bar{m}}$ is a reflection coefficient of an incoming wave $e^{2Qj\phi}$ scattering off the s-L interaction and returning as an outgoing wave $e^{-2Q(j+1)\phi}$. We are interested in a perturbation which carries no winding and no momentum in the x circle, hence we set $m = \bar{m} = 0$ and it is enough for us to note that $R_{j,0,0} \neq 0$.[28]

Since we are interested in instabilities in the interior of the space, keeping its asymptotic form fixed, then we are allowed to insert only vertex operators $\mathcal{V}$ that are normalizable (or delta-function normalizable). In the asymptotic weak coupling region we need to check:

$$\frac{1}{g_s^2}\mathcal{V} \xrightarrow{\phi \to +\infty} \text{finite.} \tag{4.34}$$

Using the asymptotic behavior and the linear dilaton slope we find two conditions for normalizability

$$2Qj + Q \leq 0 \qquad\qquad -2Q(j+1) + Q \leq 0. \tag{4.35}$$

The only solution to both constraints is $j = -\frac{1}{2} + is$ for some real value of s. This is the familiar condition that the states be delta-function normalizable in the linear dilaton

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[28]For a detailed expression for $R_{j,m,\bar{m}}$ and a comprehensive explanation of the reflection phenomena in s-L see [49].

asymptotics. From the s-L point of view it is required that the dressed $\mathcal{V}$ is marginal or relevant, while $V_{j,0,0}$ is delta-function normalizable. This imposes the following restriction on s:

$$\frac{1}{4}R_\varphi^2 - \frac{j(j+1)}{k-2} \leq 1 \quad \Rightarrow \quad R_\varphi \leq \sqrt{4 - \frac{1+4s^2}{k-2}}. \tag{4.36}$$

The smallest value of φ radius where such and interaction term is possible is at $s = 0$. Taking this value and translating back to the coordinate conventions used for AdS_3 we get

$$\beta^2 \leq \frac{4\pi^2}{k}\left(4 - \frac{1}{k-2}\right) = \beta_H^2. \tag{4.37}$$

This computation is quite suggestive. One can see many of the well known features of AdS_3 space in the s-L theory. For example, the fact one is forced to consider linear combinations of modes is much easier to interpret in the s-L language (just because there is wall from which there is reflection). In AdS_3, the need to consider linear combinations appears due to singularities of certain wave functions in the interior. In addition, the calculation itself is somewhat easier in the s-L language. Since there is a manifest weakly coupled region near the boundary, dimensions can be calculated using the zero point energy of the free theory.[29]

4.3 The regulated model

$TAdS_3$ and BTZ with pure NS-NS fields are pathological in two ways. The first is that the winding mode tachyon is delocalized and can reach the boundary. The other is that they differ by a non-normalizable mode. These pathologies are closely related to the singularities of this special point in the moduli space of the $D1-D5$ system which were discussed in [38]. Both the "long strings" and the delocalization of the tachyon correspond to the ability of long strings in thermal AdS_3 to escape to infinity.[30]

Going to the T-dual picture, we have argued that the volume of this torus is related to the asymptotic integral of the B field (which distinguishes our two phases and determines which circle is wrapped by the delocalized tachyon). In other words, this non normalizable mode is closely tied to the asymptotically flat potential for long strings. If the system is regulated in a way that all the winding states are confined to the bulk of space, than there is no measurable difference near the boundary, even not by using extended probes such as long strings. We conclude that if there exist deformations which in effect trap such long strings, $TAdS_3$ and BTZ will differ by normalizable modes and would manifestly be in the same ensemble under the usual Euclidean AdS/CFT rules.

Indeed, one can construct such examples. The $D1-D5$ CFT is singular on a subspace of its moduli space [38] and one can go off this subspace with a small deformation. We will rely on the B-field deformation constructed in [41, 42]. In the $D1-D5$ system, long

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[29]Similar calculation can also be done in AdS_3, but this requires some variable change which can actually be interpreted as T duality (see [64]).

[30]The two are actually more closely related. Both long strings in Lorentzian space and the Atick-Witten tachyon in $TAdS$ are described by a string winding one circle, and quantization on an $S^1 \times R$ worldsheet is very similar.

strings which make their way to the boundary correspond to instantonic strings on the $\mathbb{T}^4$, which is wrapped by the $D5$ branes, shrinking to zero size. This is a $D1$ brane which becomes point like and can then leave the stack. Hence, one should prevent instantons from shrinking. One way to do it is by turning on B field on the $\mathbb{T}^4$ which in effect modifies the moduli space of instantons to that of a non-commutative $\mathbb{T}^4$. In the non-commutative case, instantons cannot shrink due to the existence of a new physical scale.

Taking the near-horizon of the regulated $D1-D5$ system [41, 42] S-dualizing and Wick rotating one obtains the background

$$ds'^2_{\text{str}} = k \left[d\rho^2 + \cosh^2 \rho dt^2 + \sinh^2 \rho d\theta^2 + d\Omega_3^2 \right] + \sqrt{V} \left[\frac{(1-\epsilon)^2}{\cos^2 \gamma} dT_2'^2 + dT_2^2 \right] \quad (4.38)$$

$$e^{-2\Phi'} = g_s^2 \quad , \quad C'_{(2)} = -2 \frac{V}{g_s^2} \tan \gamma \, \varepsilon_{T_2'}$$

$$F_{(5)} = 2k \frac{\sqrt{V} \tan \gamma}{g_s^2} \frac{(1-\epsilon)^2}{\cos \gamma} (1 + \star_{10}) \left[\left(-id(\sinh^2 \rho) \wedge dt \wedge d\theta + \frac{1}{\alpha'} \varepsilon_{\Omega_3} \right) \wedge \varepsilon_{T_2'} \right]$$

$$dB'_{(2)} = 2k(1-\epsilon) \left(-id(\sinh^2 \rho) \wedge dt \wedge d\theta + \frac{1}{\alpha'} \varepsilon_{\Omega_3} \right)$$

$$dT_2'^2 = dy_1^2 + dy_2^2 \quad , \quad dT_2^2 = dy_1^3 + dy_2^4 \quad , \quad \varepsilon_{T_2'} = dy_1 \wedge dy_2$$

$$1 - \epsilon = \left(1 + \frac{V}{g_s^2} \tan^2 \gamma \right)^{-1/2} \quad , \quad 1 > \epsilon \geq 0. \quad (4.39)$$

The limit $\gamma \to 0$ (equivalently $\epsilon \to 0$) is the NS-NS Euclidean AdS_3 solution. Consequently, the dynamics of a long string is now governed by this modified Nambu-Goto action . We can expand the action at large ρ and read out the potential for the long string we studied before

$$V_{F1} \sim \int d^2\xi \left(\frac{1}{2} e^{2\rho} (1 - \alpha) + const + O(e^{-2\rho}) \right). \quad (4.40)$$

Next we compactly the angles to find a deformation of $TAdS_3$ (or $EBTZ$),

$$t \cong t + \beta \qquad\qquad \theta \cong \theta + 2\pi.$$

It is already clear that the long strings no longer have a flat potential. Rather, they are confined to the bulk of space. This was the expected effect of this deformation. Hence, small deformations of the singular point resolve the pathological non normalizable mode which is measurable at infinity. Next carry out the same set of T-dualities.[31] Writing only the NS-NS fields and focusing on the AdS_3 directions only we find the regulated cigar,

$$ds^2 = k \left[d\rho^2 + v(\rho) \left(\tanh^2 \rho d\chi^2 + d\varphi^2 \right) \right]$$

$$B'_{(2)} = 2ikv(\rho)\epsilon \, \sinh^2 \rho \left[1 + (1-\epsilon) \tanh^2 \rho \right] d\varphi \wedge d\chi$$

$$g_s' = g_s \frac{v(\rho)}{\cosh^2 \rho}$$

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[31]T-dualities with RR fields are discussed in [65, 66]. It is fortunate that in our case the RR fields has no effect in the AdS directions enabling us to use the familiar NS-NS duality formulas.

$$\varphi \cong \varphi + \beta \,,$$
$$\theta \cong \theta + 2\pi$$
$$v(\rho) = \left[1 - 2\epsilon \sinh^2 \rho + \epsilon^2 \frac{\sinh^4 \rho}{\cosh^2 \rho} \right]^{-1}. \tag{4.41}$$

We will focus on the case of β close to 2π. The technical reason is the background (4.41) has a singularity at finite ρ which is purely an artifact of the T-duality. Around $\beta \sim 2\pi$ we can truncate to leading order in ϵ. We see that the volume of the torus is rescaled by $\sim (1 + 2\epsilon \tanh^2 \rho)$. One can therefore glue, for the same background at large values of ρ, an $SL(2)/U(1) \times U(1)$ -like throat for a range of values of this volume, which implies that $TAdS_3$ and $EBTZ$ are in the same ensemble in this case.[32]

5. The middle point CFT

As discussed above, we expect (for intermediate temperatures) to find a new fixed point, where both circles are on equal footing. In this section we suggest what this theory might be, although more work is needed to verify this picture. We will mainly work in the Hawking-Page temperature (2.9), where the two circles are identical and the additional $\mathbb{Z}_2$ symmetry simplifies the discussion, but will comment on the more general case.

We concentrate on the unregulated model since we wish to use worldsheet CFT arguments. When we go to the regulated model the following changes occur

- Lines of fixed points in the unregulated model can collapse to a set of discrete points as the spacetime regulator introduces a small flow on this line. The origin of the small beta function is from the region where the unregulated model is glued to a new space which takes over as one approaches the boundary of spacetime.

- RG flow which is infinite distance in coupling space turns into a finite distance flow. The infinite distance flow is roughly similar to the flow in [11] from C/Z_N to C. Introducing the regulator is the same as focusing on the flow from C/Z_N to C in a finite region around the origin - such a region of spacetime relaxes in a finite RG distance.

With these changes in mind we can discuss both cases at the same time. The problem then boils down to the question of how one can cap, at strong coupling (or in the IR in the AdS/CFT terminology), a $linear - dilaton \times \mathbb{T}^2$ background. There are several options, which can be located at different position along the flow. In the following, we describe and evaluate these possibilities.

1. The maximally symmetric points. The maximally symmetric plane of fixed points is characterized by the fact that it has an $SL(2, \mathbb{Z})$ discrete gauge symmetry which acts on the parameters of the $\mathbb{T}^2$ (and no other parameters transform under it), i.e., other than

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[32]The dilaton needs to be adjusted at the point of the gluing which is possible. Also, the value of $\int_{\mathbb{T}^2} B$ is an irrelevant perturbation from the point of view of evolution in ρ in the cigar.

the shape of the torus the symmetry between the different directions is preserved. As usual it is natural to take the shape modulus of $\mathbb{T}^2$ for this class of theories to live in the fundamental domain of $SL(2,\mathbb{Z})$ (and of course there is still the volume modulus). The only known candidate for this theory - and one that should be considered since it has the right boundary condition to be included in the ensemble - is $Liouville \times \mathbb{T}^2$. The cap is done completely within the linear dilaton theory, and the $SL(2,\mathbb{Z})$ acts as a symmetry on a decoupled $\mathbb{T}^2$. Considering this theory, however, is problematic since, for the relevant values of the linear dilaton slope, the coefficient α in the Liouville interaction $e^{\alpha\phi}$ is imaginary and the interaction is part of the delta-function normalizable spectrum.

If such a theory can be defined, then it is defined for any value of μ_L - the coefficient of the Liouville interaction on the worldsheet. Usually, this value is considered inconsequential as it can be changed by shifting the linear dilaton coordinate ρ. However, in our case one has to keep it as a modulus of the theory since we fix the UV cut-off during a computation (at large value of ρ).[33] μ_L then measures the length of the throat from this cut-off. This gives us a line of fixed points. When fixing the boundary conditions and summing over all bulk geometries we have to sum over this line of fixed points.

It is important to emphasize that even if the CFT on the sphere is well defined for all values of μ_L, the range around $\mu_L = 0$ is problematic because then the coupling is strong near the cap and one needs to evaluate higher loops and non-perturbative effects. In particular, if the integral over μ_L diverges for $\mu_L \sim 0$, these strong coupling effects are dominant. The point $\mu_L = 0$ itself - i.e. $linear - dilaton \times \mathbb{T}^2$ - certainly does not exist as a perturbative String background. One can estimate whether the behavior near $\mu_L \sim 0$ is indeed problematic, since then the throat is long (even in the regulated model) and the μ_L dependence of correlation functions dominated by the throat is governed by a KPZ scaling similar to (4.12). We leave this to future work.

The suggestion here is in the same spirit as in [67], which treats the Horowitz-Polchinski correspondence principle [68, 69] from a worldsheet point of view (for a specific class of black holes). In that case, the near horizon of a black hole, in the vicinity of the black hole/excited string phase transition, is described by an $SL(2)/U(1)$ cigar, and all angular information about the black hole disappears. Here we also obtain a similar throat but keep an additional S^1 which can encode angular information (since the chemical potential is encoded in the shape of the $\mathbb{T}^2$).

2. A partially symmetric point. In the language of the Euclidean instantons described in section 2.1 the transformation that interchanges $EBTZ$ and $TAdS_3$, is $\tau' = -\frac{1}{\tau}$. At the Hawking page temperature $\tau = i$ and the transformation is a $\mathbb{Z}_2$ symmetry of the boundary. The symmetry is broken in the interior of space by the choice of which circle contracts. One can ask whether there is a symmetric point which respects this $\mathbb{Z}_2$ symmetry.

Our conjecture for the $\mathbb{Z}_2$ symmetric point is to take (4.30) and set $\lambda_1 = \lambda_2$ (remember that $R_x = R_\phi$ in this point of moduli space), i.e.,

$$\lambda e^{b\phi} \left(\cos R(x_L - x_R) + \cos R(\varphi_L - \varphi_R) \right), \tag{5.1}$$

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[33]Phrased in another way, we fix the value of g_s at the cut-off point.
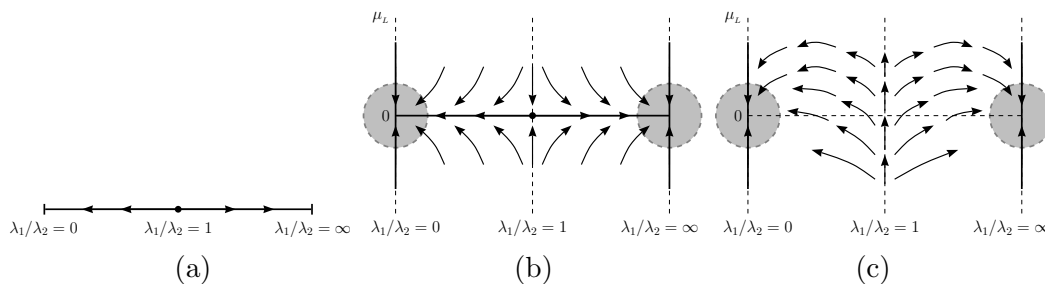
Figure 4: The three possible flow patterns at the HP temperature. Figure (a) is the naive one dimensional flow in $\frac{\lambda_1}{\lambda_2}$. Figure (b) is the conjectured flow case when we include a small μ_L. Figure (c) is an example of what could go wrong. In the latter $\lambda_1 = \lambda_2$ is not a fixed point

where b is determined by fixing the scaling dimension and R is the radius obtained using FZZ duality. $\mathrm{SL}(2, \mathbb{Z})$ is still a discrete gauge symmetry but now it acts on the shape modulus of the $\mathbb{T}^2$ and on the two cycles on which we chose to turn on the s-L interaction. The fundamental domain of this theory is now larger than the fundamental domain of $\mathrm{SL}(2, \mathbb{Z})$.

We would like to emphasize again that this is a conjecture, as it is not clear that this theory exists. For examples the techniques of [48] cannot be applied in any simple way. Hence, we will have to argue its existence indirectly as follows.

If we can restrict our attention to the two operators in (4.30), which we can for $\lambda_1 \ll \lambda_2$ (or $\lambda_2 \ll \lambda_1$), then the existence of an unstable fixed point at $\lambda_1 = \lambda_2$ is guaranteed. The flow is then depicted in figure 4a. However, it is not clear that one can restrict to only these two operators. If we follow [48] then there are at least three operators which play a role. In addition to the two sine-Liouville interactions in (4.30), one also expects that a Liouville interaction could be turned on (we will denote the coefficient of the latter μ_L here as well). If $\lambda_1 = 0$ (or $\lambda_2 = 0$) FZZ duality conjectures that we can set $\mu_L = 0$ (as was shown explicitly for $k = 9/4$ in [48]). For μ_L very small, such that the Liouville cap is behind the sine-Liouville i.e. deeper into the strong coupling area, we expect μ_L to be irrelevant. This describes the shaded areas in figure 4b. Note however that close to the entire $\mu_L = 0$ line (i.e. $\mu_L \ll max(\lambda_1, \lambda_2)$) the Liouville interaction cap is behind the sine-Liouville combined cap. Hence we expect that the Liouville interaction will always be irrelevant for small enough μ_L, which gives the entire 4b phase diagram and an unstable fixed point at $\lambda_1 = \lambda_2$, $\mu_{\text{Liouville}} = 0$.

These arguments implicitly assume that certain orders of limit do not matter (for example if we follow [48], which is applicable only for $k = 9/4$ to start with, where one starts with small λ and resums the expansion in this parameter) and that no additional operators can appear in front of the s-L caps. Hence it is suggestive, but not a rigorous argument.

It is also interesting to see what other middle point CFTs can be obtained by using other caps - i.e. by taking, in the asymptotic *linear dilaton* $\times \mathbb{T}^2$ some other operators from the $\mathbb{T}^2$ and dressing them by an operator from the linear dilaton, which grows in the

strong coupling region. The most natural example would be to take the Poincaré patch.[34] We begin with the following Euclidean metric, which is a solution of GR,

$$ds^2 = k\left(d\rho^2 + \frac{e^{2\rho}}{4}d\theta^2 + \frac{e^{2\rho}}{4}dt^2\right)$$

$$B_{(2)} = -\frac{ik}{2}\left(e^{2\rho} - 2\right)dt \wedge d\theta$$

$$e^\Phi = g_s$$

$$t \cong t + \beta, \quad \theta \cong \theta + 2\pi. \tag{5.2}$$

In the above we kept the constant term in the B-field, which plays an important role in our backgrounds, as is clear by now. This solution has the same symmetry pattern of the Liouville theory. We have chosen this specific B field since it is the one that corresponds to the Poincaré patch. However, as far as the behavior at infinity is concerned we could change this value (while staying in the ensemble or in the regulated model).

Next we would like to go over the *linear dilaton* $\times$ $\mathbb{T}^2$ picture. Carrying out the sequence of T-dualities as before we find

$$ds^2 = k\left(d\rho^2 + \frac{d\chi^2 + d\varphi^2}{1 + e^{-2\rho}}\right)$$

$$B_{(2)} = \frac{2ik}{(1 + e^{2\rho})}d\varphi \wedge d\chi$$

$$\Phi = \log g_s + \log\frac{4}{1 + e^{2\rho}}$$

$$\varphi \cong \varphi + \beta, \quad \chi \cong \chi + 2\pi. \tag{5.3}$$

The geometry far away at the weak coupling $\rho \to \infty$ region indicates a cap made out of the volume and B-field of the $\mathbb{T}^2$, dressed by a profile in the Liouville direction. At $\rho \to -\infty$ the two circles shrink while the B-field and coupling go to a constant value.

Since the circles shrink exponentially, it is not clear how to analyze this theory. We would like however to point to the possibility that this theory flows to a Liouville theory. The shrinking $\mathbb{T}^2$ could either disappear from the theory (as is the case in [67] for an S^2) or it can stabilize at some finite stringy radii (since here, unlike S^2, there is a CFT for any radii of $\mathbb{T}^2$) but in any case, from considering the central charge, a linear dilaton term should be generated. The perturbation at infinity then has the right quantum numbers to mix into a Liouville wall. We can perhaps separate the Liouville wall from where the $\mathbb{T}^2$ shrinks by changing the value of the B-field.

We have presented our solutions only at the HP temperatures, but similar caps can occur for other values of the temperature as well. This allows testing our proposal by approaching the Hagedorn temperature of AdS_3 from below. As in figure 21, in this regime the two fixed points should be close to each other, and one can hope to perform a perturbative calculation detecting the nearby fixed point close to AdS_3. This would be especially interesting in the s-L language, where one can test these ideas explicitly.

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[34]This analysis was also carried out by S. Minwalla.
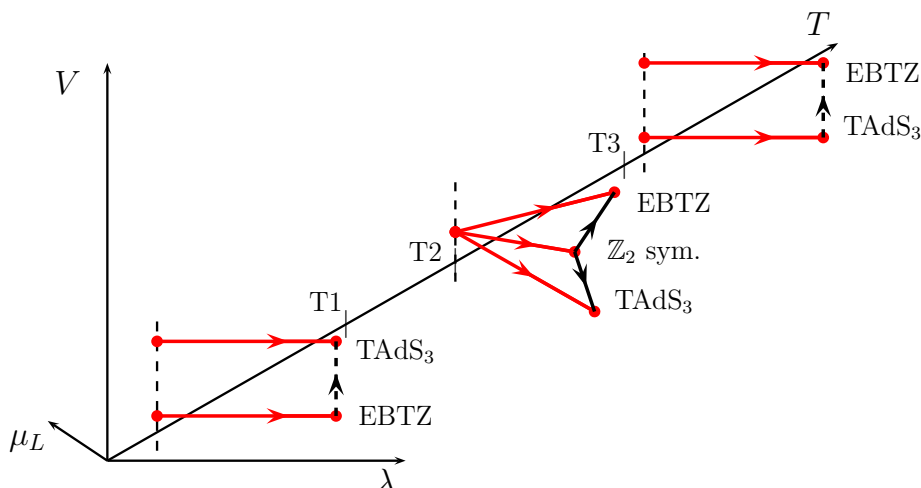
Figure 5: Qualitative picture of the flow between the various theories

We should mention that the role of the unstable fixed point in the case of AdS_5 is played by the small black hole which can decay to a larger black hole (which is stable at high temperatures) or a gas of particles in AdS_5 (which is stable at low temperatures). In some sense, our solution is a small black hole as well since the time circle is contractible and the horizon has exactly zero area.

5.1 Summary of the flows

Let us briefly summarize the flows.[35] Part of this picture can be derived from considering the free energy in the dual field theory, but now one can be more precise about the different CFT's in the different regimes.

The qualitative flow between the CFT's is described in figure 5.

In this diagram the axis that goes into the page is the temperature axis. The four temperatures regimes, separated by temperatures $T1$, $T2$ and $T3$ refer to those in figure 1. This diagram applies for the unregulated model, and we discussed before how to pass from it to the flow picture for the regulated model.

The relevant axis in the diagram, in addition to the temperature axis, are λ which stands for both sine-Liouville interactions, V which denotes the volume of the $\mathbb{T}^2$ at infinity and μ_L which is the coefficient of the Liouville interaction. We will be interested in the regime where the Liouville wall is behind the s-L walls, hence we can set $\mu_L = 0$ in the discussion. There are 3 types of arrowed lines - red arrowed lines denote using a marginal s-L interaction to deform the theory (and FZZ duality to write it geometrically). This is not really a flow but a finite distance change as far as the worldsheet CFT is concerned. Solid black arrowed lines denote ordinary CFT flow. Dashed black arrowed lines denote an infinite distance flow which can also change the boundary conditions on the volume of the

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[35] A boundary CFT interpretation of the phases in AdS_3 was proposed in [70].

$\mathbb{T}^2$. Vertical unarrowed dashed lines simply indicate lines of different V, keeping all other parameters fixed.

The figure maps the different theories at 3 different temperatures: The HP temperature, at a high temperature (above the Hagedorn temperature) and at a low temperature (below the dual Hagedorn temperature, where the $EBTZ$ is tachyonic). The latter two are of course images of each other under the exchange of $EBTZ$ with $TAdS_3$ and β with inverse β.

At the HP temperature we can turn on a s-L interaction on either circles to go either $EBTZ$ or $TAdS_3$. We also expect that there is a symmetric point where both caps are turned on. From it, one can flow to either $EBTZ$ or $TAdS_3$.

At temperatures above the Hagedorn temperature we can go to either $EBTZ$ or $TAdS_3$ using an s-L interaction, but one needs to start with different values of V. This is depicted by the red line starting at different values of V at $\lambda = 0$. The flow from $TAdS_3$ to $EBTZ$ via the Atick-Witten tachyon has to change the value of V.

Tachyons can change the boundary conditions of non-compact CFTs under some circumstances. The most straightforward case are tachyons which are completely delocalized. In this case one can condense their zero mode (the most familiar case is that of the bosonic theory) and the entire space changes, including the behavior at infinity. Such a flow is expected to change the central charge by the Zamolodchikov c-theorem [22]. The other case is if they are localized but mix into the delta function normalizable states of the non-compact CFTs. In this case a shell expands from where the tachyon is localized to the boundary but, strictly speaking, it does not reach it in finite distance in coupling space. This is what happens in [11] and in such cases the central charge does not have to change.

The situation here is intermediate. As we saw before we can identify the tachyon in the delta function normalizable spectrum in $sine - Liouville \times S^1$. If we build a wave packet out of it then we have a kind of localized tachyon, which can mix into the delta function normalizable states of the volume mode (which are not necessarily tachyonic). The latter will make up the shell that propagates to infinity of spacetime and eventually changes, in infinite distance in coupling space, the asymptotic volume of the $\mathbb{T}^2$. This picture is supported by the regulated model where the flow is a localized tachyon, finite distance, flow which changes only a region of space (very similar to $AdS_5 \times S^5$). As the regulator is removed the length of the region which changes grows to infinity until it covers the entire space suggesting an infinite distance flow in which the original tachyon mixes with the modes described above of the volume.

Not building a wave packet but condensing the $s = 0$ mode is more puzzling (see the discussion around (4.36)), and we postpone a full discussion to future work. Let us point out however that it is not clear that discussing only this mode makes sense. In situations where a delta-function normalizable spectrum of tachyonic operators arise due to a translation invariance in a non-compact direction one can safely discuss the condensation of the zero mode because then the non-compact direction decouples and one discusses a discrete operator in the remaining CFT. This is not the case here because there is no translation symmetry in ρ and the tachyon profile is ρ dependent. In fact, if one wants to take the back-reaction of this field (or any wave packet) then the back-reaction is suppressed

at large values of ρ due to the decrease in g_s suggesting again the localized wave packet construction.

6. Discussion and open questions

We suggested a unified description for geometric and tachyonic capping in String theory, and applied it to the study of the $TAdS_3/EBTZ$ phase diagram where it turns out to be a useful tool for understanding the intermediate worldsheet CFTs. We expect that this technique of evaluating competing tachyon condensation, would be useful in other topology change problems in String theory and GR. Here we used the symmetries of the problem, such as the $\mathbb{Z}_2$ symmetry which interchanges the two circles, to argue the existence of a new fixed point. But more generally one expects similar fixed points at the end of the separatrix that separates between the phase in which one tachyon dominates to the phase in which another does. This entails the comparison of different "walls" made out of Liouville-like interactions times different operators from some internal compact CFTs.

A concrete open problem is to provide more evidence for the existence of the new conformal field theory which we conjectured to describe the unstable phase. There are many tests one can make to check this conjecture, some of which were described in section 5. Another open problem is that the hard dynamical part of the process - i.e., the details of the RG flows - remains to be understood. Particularly interesting are the flows away from the HP temperature due to the need to change a non-normalizable mode. This problem occurs in the theory because the spacetime CFT is singular, and hence the UV/IR relation in $AdS_{3,\text{NS-NS}}$ is rather unusual. In the regulated model, in which RR charges are turned on, the UV/IR relation is the standard one, but this model is not solvable. We plan to pursue these checks in the future.

This project started when attempting to study the Lorentzian BTZ as a global model which realizes the time dependent Misner space or Grant space. However, already the Euclidean $BTZ/TAdS_3$ exhibits a set of poles in the partition functions which are practically identical to the poles encountered in Misner/Grant spaces [72, 73]. In AdS_3 these poles disappear when turning on a RR fields - it would be very interesting to find out what is the corresponding deformed model in the Misner/Grant cases.

Acknowledgments

A. Mini-Superspace analysis of the winding string

The Atick-Witten tachyon discussed in section 3 is a winding mode around the Euclidean time circle [10]. We apply a minisuperspace quantization of the string in thermal AdS_3 to study the properties of this mode. We use bosonic String theory and ignore the existence of the usual bulk tachyon in the spectrum. The action of a string in thermal AdS_3 is,

$$S = \frac{k}{4\pi} \int d^2\sigma \left[(\partial_a \rho)^2 + \cosh^2 \rho (\partial_a t)^2 + \sinh^2 \rho (\partial_a \theta)^2 - 2 \sinh^2 \rho \, \epsilon_{ab} \partial_a t \partial_b \theta \right], \qquad \text{(A.1)}$$

where a and b are worldsheet vector indices, ϵ_{ab} is a 2-dim antisymmetric tensor. The target space coordinates match those of (4.14). We consider a minisuperspace ansatz[36] for the winding mode,

$$t = t(\sigma^2) + \frac{w\beta}{2\pi}\sigma^1 \qquad \rho = \rho(\sigma^2) \qquad \theta = \theta(\sigma^2) \,, \qquad w \in \mathbb{Z}. \qquad \text{(A.2)}$$

The minisuperspace Lagrangian is,

$$S = \frac{k}{2} \int d\sigma^2 \left[\dot{\rho}^2 + \cosh^2(\rho)\dot{t}^2 + \sinh^2(\rho)\left(\dot{\theta} - \frac{w\beta}{2\pi}\right)^2 + \left(\frac{w\beta}{2\pi}\right)^2 \right], \qquad \text{(A.3)}$$

the dot stands for derivatives in respect of the worldsheet coordinate σ^2. A canonical quantization of the Euclidean action follows [71] (chapter 8),

$$v^n = i\dot{X}^n \qquad\qquad P_n = -\frac{\partial \mathcal{L}}{\partial v^n} \qquad\qquad H = \mathcal{L} + P_n v^n.$$

Applied to the minisuperspace action (A.3) for winding $w = 1$ we find the Hamiltonian[37]

$$H = \frac{1}{2k}\left(-\partial_\rho^2 - \frac{\partial_\rho\left(\sqrt{g}\right)}{\sqrt{g}}\partial_\rho \right) + \frac{(P_t)^2}{2k\cosh^2\rho} +$$
$$+ \frac{1}{2k\sinh^2\rho}\left(P_\theta + \frac{ik\beta}{2\pi}\sinh^2\rho \right)^2 + \frac{k}{2}\cosh^2\rho\left(\frac{\beta}{2\pi}\right)^2 + \frac{a}{2k}, \quad \text{(A.4)}$$

with $\sqrt{g} = \frac{1}{2}\sinh(2\rho)$. The additional constant a comes by computing correctly the zero point energy of all modes that have been integrated out (this will be carried out in the sequel). The imaginary term in the Hamiltonian is a consequence of the imaginary B-field. For the zero momentum case $P_\theta = P_t = 0$ the imaginary part vanishes and the eigenvalue equation simplifies. The regularity of the wave function on the disc ρ, θ imposes a simple constraint at $\rho = 0$

$$\frac{\partial \Psi(\rho)}{\partial \rho}\Big|_{\rho=0} = 0, \qquad\qquad\qquad\qquad\qquad \text{(A.5)}$$

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[36] It is easy to check that the ansatz is competent with the equation of motion.

[37] It is important to take care of ordering ambiguities in the Hamiltonian. In the case at hand, all ambiguities are fixed by the existence of a unique quadratic differential consistent with the symmetries.

and the eigenvalue problem is

$$E\Psi(\rho) = \frac{1}{2k}\left[-\partial_\rho^2 - 2\coth(2\rho)\partial_\rho + \left(\frac{\beta k}{2\pi}\right)^2 + a\right]\Psi(\rho). \tag{A.6}$$

It is interesting that the potential energy for this winding mode is exactly constant (due to cancelation of the winding energy by the B-field coupling). The solutions of this eigenvalue problem are well known. Their asymptotic form and the exact eigenvalue set are[38]

$$\Psi_j(\rho) \sim \exp\left(2j\,\rho\right) \qquad\qquad E_j = \frac{1}{2k}\left[-4j(j+1) + \left(\frac{\beta k}{2\pi}\right)^2 + a\right]. \tag{A.7}$$

Remembering that there is a measure factor $\sqrt{g} \sim e^{2\rho}$ in the norm formula, we arrive at the conclusion that the allowed set of j's is $j = -\frac{1}{2} + is$ where s is any real number. For other cases the wave function is either not normalizable at infinity or singular at the origin. Thus only continuum normalizable solutions exist (which in particular means that these winding modes are not strictly localized). To reproduce the Hagedorn temperature as well as the relevant terms in the one loop partition function, we need to compute a, which we do in the following subsection.

A.1 Calculation of the zero point constant

To perform this computation one should examine more carefully the structure of the full CFT, using current algebra techniques. Following [39], the worldsheet stress tensor of H_3^+ is expressed in terms of affine $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ algebra currents,

$$T^{ws}(z) = \frac{1}{k-2}\left(-(J^3)^2 + J^+J^-\right) \qquad \bar{T}^{ws}(\bar{z}) = \frac{1}{k-2}\left(-(\bar{J}^3)^2 + \bar{J}^+\bar{J}^-\right). \tag{A.8}$$

The currents obey the OPE,

$$
\begin{aligned}
J^3(z)J^\pm(w) &\sim \frac{\pm J^\pm(w)}{z-w} \\
\bar{J}^3(\bar{z})\bar{J}^\pm(\bar{w}) &\sim \frac{\pm \bar{J}^\pm(\bar{w})}{\bar{z}-\bar{w}} \\
J^-(z)J^+(w) &\sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w} \\
\bar{J}^-(\bar{z})\bar{J}^+(\bar{w}) &\sim \frac{k}{(\bar{z}-\bar{w})^2} + \frac{2\bar{J}^3(\bar{w})}{\bar{z}-\bar{w}} \\
J^3(z)J^3(w) &\sim \frac{-k/2}{(z-w)^2} \\
\bar{J}^3(\bar{z})\bar{J}^3(\bar{w}) &\sim \frac{-k/2}{(\bar{z}-\bar{w})^2}.
\end{aligned}
\tag{A.9}
$$

$TAdS_3$ with parameter τ is an orbifold of H_3^+. The orbifold generators twist the currents

$$\left(J^3,\ J^+,\ J^-\right) \longrightarrow \left(J^3,\ e^{-2\pi i\tau}J^+,\ e^{+2\pi i\tau}J^-\right). \tag{A.10}$$

---

[38]We ignore the shift in k which is invisible in the simple reduction of the model we employ here.

These monodromy conditions allow us to write the oscillator expansion of the currents in the n-th twisted sector

$$J^3(z) = \sum_m \frac{J_m^3}{z^{m+1}} \qquad\qquad J^\pm(z) = \sum_m \frac{J_{m\mp n\tau}^\pm}{z^{m+1\mp n\tau}}. \qquad (A.11)$$

Using standard CFT techniques, we can derive the twisted commutation relations

$$[J_m^3, J_{l\mp n\tau}^\pm] = \pm J_{m+l\mp n\tau}^\pm$$
$$[J_{m+n\tau}^-, J_{l-n\tau}^+] = 2J_{m+l}^3 + \delta_{m+l} k(m+n\tau)$$
$$[J_m^3, J_l^3] = -m\frac{k}{2}\delta_{m+l}. \qquad (A.12)$$

From here, one has all the information needed to compute the minisuperspace Hamiltonian with the correct zero point energy. The ambiguity previously present is now resolved by calculating the normal ordering of all the higher string modes and regularizing the infinite sum in a way consistent with the Virasoro algebra. We list the contributions to the total zero point energy in the following:

- ghosts contribute 2/12

- the unitary CFT $\mathcal{M}$ adds $\frac{1}{12}\left(\frac{6}{k-2} - 23\right)$

- from the Klein-Gordon equation (zero modes) we get $\frac{1}{2(k-2)}$

- ordering of higher modes (using zeta function regularization) $-\frac{k}{4(k-2)}$

- the length of the string gives $\frac{1}{2}k(\beta/2\pi)^2$

Summing up all the contribution we find the zero point energy

$$\frac{1}{2}k(\beta/2\pi)^2 - 2 + \frac{1}{2(k-2)}. \qquad (A.13)$$

This vanishes exactly at the Hagedorn temperature (3.4). One can do a little more matching (the zero mode) pieces of the exact partition function (3.1) by calculating the partition function of the minisuperspace model

$$\sum e^{-2\pi\tau_2 H} = \sum_{m\in\mathbb{Z}} \int_{s\in\mathbb{R}} ds\, e^{-2\pi i\tau_2 m\beta} e^{-2\pi\tau_2 \frac{2s^2}{k-2}} e^{-2\pi\tau_2\left(\frac{1}{2}k\beta^2 - 2 + \frac{1}{2(k-2)}\right)}. \qquad (A.14)$$

The poles are reproduced from the sum over imaginary energies, as happens in some time dependent backgrounds [73] (see [74] for introduction to these time dependent models).The power $\frac{1}{\sqrt{\tau_2}}$ in the partition function is a consequence of summing over a continuum of states, as expected.

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